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A critical state two-surface plasticity model (Manzari & Dafalias, 1997) in OpenSees

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Outline of the constitutive model

Development of plasticity models that accurately simulate the behavior of engineering materials has been of great interest to engineers in recent years. In the case of geo-materials such as granular soils, the mathematical formulation of appropriate plasticity models is rather complex. It requires pressure sensitivity of the elastic bulk and shear moduli, third stress invariant dependence and in some cases two or multi-surface plasticity formulations. Moreover, to be useful in engineering calculations these complex models require efficient and robust numerical implementation.

In this research an advanced critical state plasticity model developed by Manzari and Dafalias [1] has been employed in the numerical analyses. This model is based on the following characteristics of the stress-strain behavior of sands: (i) the strength and volume change behavior of granular soils are governed by the combined effect of density (void ratio) and confining stress. This combined effect is often represented by state parameter, $\psi = e - e_c$, where e_c is the critical void ratio corresponding to the existing confining stress on the soil element. (ii) when sheared in monotonic loading, granular soil that is denser than critical will exhibit a peak strength and upon further shearing a softening regime will appear in the stress-strain relationship. Granular soils with a void ratio less than their critical void ratio show a prevalent contractive response upon shearing toward critical state. A schematic representation of the two-surface model in the π-plane is shown in figure 1.

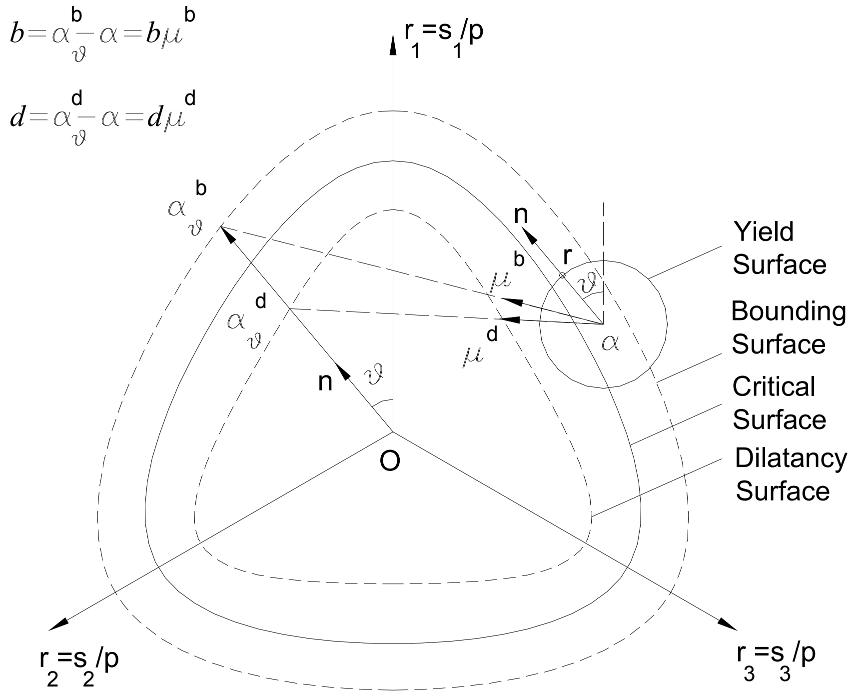

Figure 1. Schematic representation of the two-surface model in π-plane [1]

The formulation of the model is based on the general two-surface plasticity and the bounding surface plasticity theory. The state parameter, ψ, is used as the key ingredient to accurately model the effect of critical state for sands. The numerical efficiency of the model is good because only the yield surface must be updated owing to kinematic and isotropic hardening. The other surfaces are fully determined by the value of state parameter ψ. A brief description of the model is given below [1,2]:

Elastic moduli

The elastic moduli, K and G, are defined through the following standard relationships:

$$\dot{\boldsymbol{\varepsilon}}_q^e=\frac{\dot{\mathbf{s}}}{2G}\ ,\ \dot{\varepsilon}_v^e=\frac{\dot{p}}{K} \qquad (1.)$$

$\dot{\boldsymbol{\varepsilon}}_q^e$ and $\dot{\varepsilon}_v^e$ are the elastic components of deviatoric and volumetric strain increments respectively and $\dot{\mathbf{s}}$ and $\dot{p}$ are deviatoric and mean effective parts of stress increment tensor. The isotropic hypoelasticity assumption is adopted, giving:

$$K = K_0\left(\frac{p}{p_{at}}\right)^a, \; G = G_0\left(\frac{p}{p_{at}}\right)^a \tag{2.}$$

where p_{at} is the atmospheric pressure used as a reference pressure, for which $K = K_0$ and $G = G_0$, and a is a properly defined exponent yielding the variation of G and K with p.

Yield function

The yield surface is a cone with circular cross section in π plane:

$$F(\boldsymbol{\sigma}, \boldsymbol{\alpha}, m) = \sqrt{(\mathbf{s} - p\boldsymbol{\alpha}) : (\mathbf{s} - p\boldsymbol{\alpha})} - \sqrt{2/3}mp = 0 \tag{3.}$$

p and m are mean effective stress and size of the yield surface respectively, $\mathbf{s}$ is the deviatoric stress tensor and $\boldsymbol{\alpha}$ is the back-stress ratio deviatoric tensor which determines the position of the axis of the cone.

The normal to the yield surface, defining the loading direction, is given as

$$\mathbf{L} = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{n} - \frac{1}{3}N\mathbf{I} \tag{4.}$$

where $\mathbf{n} = \dfrac{\mathbf{r} - \boldsymbol{\alpha}}{\sqrt{2/3}m}$, $N = \boldsymbol{\alpha} : \mathbf{n} + \sqrt{2/3}m = \mathbf{n} : \mathbf{r}$, $\mathbf{r} = \dfrac{\mathbf{s}}{p}$ and $\mathbf{I}$ is the second rank identity tensor.

Flow rule

The plastic strain component is given by:

$$\dot{\boldsymbol{\varepsilon}}^{p} = \langle\lambda\rangle\frac{\partial Q}{\partial\boldsymbol{\sigma}} = \langle\lambda\rangle\left(\mathbf{n} + \frac{1}{3}D\mathbf{I}\right) \tag{5.}$$

where Q is the plastic potential surface and D the dilatancy coefficient. In general $D \neq -N$, which leads to a 'non-associated flow rule' ($F \neq Q$). λ is the loading index enclosed by the Macauley brackets $\langle\ \rangle$ to indicate loading/unloading/neutral loading and is defined by:

$$\lambda = \frac{1}{K_p}\left(\frac{\partial F}{\partial\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}\right) = \frac{2G\mathbf{n} : \dot{\boldsymbol{\varepsilon}}_q - KN\dot{\varepsilon}_v}{K_p + 2G - KDN} \tag{6.}$$

where K_p is the plastic modulus.

Stress-rate expression

Using the additive decomposition one can find the increment of stress in terms of total strain rates of $\dot{\boldsymbol{\varepsilon}}_q$ and $\dot{\varepsilon}_v$ as:

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= \dot{\mathbf{s}} + \dot{p}\mathbf{I} = 2G\dot{\boldsymbol{\varepsilon}}_q^e + K\dot{\varepsilon}_v^e\mathbf{I} \\ &= 2G(\dot{\boldsymbol{\varepsilon}}_q - \dot{\boldsymbol{\varepsilon}}_q^p) + K(\dot{\boldsymbol{\varepsilon}}_v - \dot{\boldsymbol{\varepsilon}}_v^p)\mathbf{I} \\ &= 2G\dot{\boldsymbol{\varepsilon}}_q + K\dot{\varepsilon}_v\mathbf{I} - \langle\lambda\rangle(2G\mathbf{n} + KD\mathbf{I})\end{aligned} \tag{7.}$$

Bounding surface and dilatancy surface

The critical state of granular soil and its tendency for volume change are highly dependent on the direction of the stress path to which the soil is subjected. This can be achieved using the modified Lode angle θ as:

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\left(\frac{\bar{S}}{\bar{J}}\right)^3 \tag{8.}$$

in which:

$$\bar{J} = \left[\frac{1}{2} tr\bar{\mathbf{r}}^2\right]^{1/2}, \ \bar{S} = \left[\frac{1}{3} tr\bar{\mathbf{r}}^3\right]^{1/3}, \ \bar{\mathbf{r}} = \mathbf{r} - \boldsymbol{\alpha} \tag{9.}$$

The bounding surface, dilatancy surface, and critical surface are defined in the following forms:

$$\boldsymbol{\alpha}_{\theta}^{\mathrm{b,c,d}} = \sqrt{(2/3)}\alpha_{\theta}^{a}\mathbf{n} \tag{10.}$$

$$\alpha_{\theta}^{b} = g(\theta, c)M_c + g(\theta, c_b)k_c^b\langle -\psi \rangle - m \tag{11.}$$

$$\alpha_{\theta}^{d} = g(\theta, c)M_c + g(\theta, c_d)k_c^d\psi - m \tag{12.}$$

$$\alpha_{\theta}^{c} = g(\theta, c)M_c - m \tag{13.}$$

The above equations involve M_c (the critical stress ratio) and the two model parameters, k_c^b and k_c^d, that are used to define the bounding and dilatancy surfaces on the compression side. In order to complete the definition of these surfaces, it is necessary to define M_e, k_e^b, and k_e^d that are the corresponding values on the extension side of these surfaces. Parameters c, c_b and c_d define the ratios between the values on the compression side of the above-mentioned surfaces to those on the extension side of those surfaces ($c = M_e/M_c$, $\mathrm{c_b} = k_e^b / k_c^b$, $\mathrm{c_d} = k_e^d / k_c^d$). The equation for $g(\theta, c)$ is chosen as $g(\theta, c) = 2c/\left((1+c) - (1-c)\cos 3\theta\right)$.

In the above equations, state parameter, ψ, is used to incorporate the critical state of the sand in its stress-strain behavior. Considering a straight-line approximation for the $e_c - \ln p$ relationship e_c is defined as $e_c = (e_c)_{ref} - \lambda \ln\left(p/p_{ref}\right)$. Here λ is the slope of the critical state line in the $e_c - \ln p$ plane and $(e_c)_{ref}$ is the critical void ratio corresponding to a reference pressure, p_{ref}.

Hardening laws

Both isotropic and kinematic hardenings are used in the model. The evolution equations for the size, m, and the back stress, a, are given as:

$$\dot{m} = -c_m \dot{e}^p = \langle\lambda\rangle c_m (1+e_0) D = \langle\lambda\rangle \overline{m} \tag{14.}$$

$$\dot{\boldsymbol{\alpha}} = \langle\lambda\rangle h(\boldsymbol{\alpha}_\theta^{\mathrm{b}} - \boldsymbol{\alpha}) = \langle\lambda\rangle h\mathbf{b} = \langle\lambda\rangle hb\boldsymbol{\mu}^{\mathrm{b}} = \langle\lambda\rangle \overline{\boldsymbol{\alpha}} \tag{15.}$$

The c_m is a model parameter and $\dot{e}^p = -(1+e_0)\dot{\varepsilon}_v^p$ is the plastic rate of change of void ratio. The tensor $\boldsymbol{\alpha}_\theta^b$ indicates the image of the current stress state on the bounding surface. The tensor $\boldsymbol{\mu}^b$ is a unit tensor in the direction of $\mathbf{b} = \boldsymbol{\alpha}_\theta^b - \boldsymbol{\alpha}$ which is defined in figure 1 as the vector (actually a second rank tensor) connecting the current back stress to its image on the bounding surface, $\boldsymbol{\alpha}_\theta^b$. The function h in the evolution equation for back stress, $\boldsymbol{\alpha}$, is chosen based on the original proposition by Dafalias and Popov [???] for two-surface models, that is:

$$h = h_0 \frac{|\mathbf{b}:\mathbf{n}|}{b_{ref} - |\mathbf{b}:\mathbf{n}|} \tag{16.}$$

In this equation, h_0 is a positive model parameter and b_{ref} is a reference value of $\mathbf{b}$ which is chosen as:

$$b_{ref} = 2\sqrt{(2/3)}\alpha_c^b \tag{17.}$$

Dilatancy coefficient

The dilatancy coefficient, D, is related to the distance from the dilatancy surface $\mathbf{d}$:

$$D = A\ (\boldsymbol{\alpha}_\theta^{\mathrm{d}} - \boldsymbol{\alpha}) : n = A\mathbf{d}:\mathbf{n} = Ad\boldsymbol{\mu}^{\mathrm{d}} : \mathbf{n} \tag{18.}$$

A is a positive model parameter. The tensor $\boldsymbol{\alpha}_{\theta}^{d}$ indicates the image of the current stress state on the dilatancy surface (as defined earlier). The tensor $\boldsymbol{\mu}^{d}$ is a unit tensor in the direction of $\mathbf{d} = \boldsymbol{\alpha}_{\theta}^{d} - \boldsymbol{\alpha}$ and $\mathbf{n}$ is the deviatoric part of the unit normal to the yield surface at the current stress state (Figure 1).

The plastic modulus

It is now possible to have a specific expression for plastic modulus using the definitions of $\overline{m}$, $\overline{\boldsymbol{\alpha}}$ and D:

$$K_p = -\frac{\partial F}{\partial \boldsymbol{\alpha}} : \overline{\boldsymbol{\alpha}} - \frac{\partial F}{\partial m} : \overline{m} = p\left(h\mathbf{b} : \mathbf{n} + \sqrt{2/3}c_m (1+e_0) A\mathbf{d} : \mathbf{n}\right) \tag{19.}$$

Implementation of elastic-plastic material model in OpenSees

OpenSees uses a unified frame to implement the incremental elastic-plastic material models which usually consist of the three main elements: yield surface (function), plastic flow direction (from the potential function), and the hardening-softening evolution laws. This concept is so called template elastic-plastic computations in geomechanics [4].

1. Elastic-plastic material model

The general elements of elastic-plastic material models for geomaterials can be concluded as: elasticity, yield surface, plastic flow direction, and the hardening-softening evolution laws.

1) Elasticity

The simplest form of elasticity is:

$$\Delta\sigma_{ij} = E_{ijkl}\Delta\varepsilon_{kl} \tag{20.}$$

where E_{ijkl} is the fourth order elastic stiffness tensor with 81 components in total. For isotropic elasticity, E_{ijkl} has the general form as:

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{21.}$$

where λ and μ are *Lame coefficients*:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},\ \mu = \frac{E}{2(1+\nu)}$$

and E and ν are *Young's modulus* and *Poisson's ratio*, respectively.

In Manzari & Dafalias model (1997), E is assumed to be a function of stress state and ν is assumed to be a constant.

$$E = E_o\left(\frac{P}{P_{atm}}\right)^{\alpha},\ \nu = const. \tag{22.}$$

2) Yield function

A typical form of yield function is:

$$F(\sigma, q_*) = 0 \tag{23.}$$

where σ and q_* are stress tensor and internal variables, respectively. The stress state inside the yield surface F is assumed to be elastic. Stress state on the yield surface is assumed to generate plastic deformation. In Manzari & Dafalias model (1997), the yield surface is a very small cone.

3) Plastic flow direction

The plastic flow directions are usually derived from potential function. If potential function is assumed to be the same as yield function, it is so called associated flow rules. If potential function is different from yield function, it is so called non-associated flow rules. In Manzari & Dafalias model (1997), non-associated flow rule is assumed.

4) Hardening-softening evolution laws

The change in size or/and shape of yield surface and potential surface is controlled by hardening-softening evolution laws. Depend on the types (size or shape) to be controlled, evolution laws can be divided into two isotropic and kinematic. In Manzari & Dafalias model (1997), the size of yield surface is controlled by a scalar internal variable *m*, and the shape is controlled by a tensor internal variable α - back-stress ratio deviatoric tensor.

2. Elastic-plastic formulation

The general constitutive relations of elastic-plastic material can be characterized as:

$$d\varepsilon = d\varepsilon^{el} + d\varepsilon^{pl} \tag{24.}$$

$$d\sigma = E_{ijkl} d\varepsilon^{el} \tag{25.}$$

$$d\varepsilon^{pl} = \lambda \frac{\partial Q}{\partial \sigma} = \lambda m(\sigma, q_*) \tag{26.}$$

$$dq_* = \lambda h_*(\sigma, q_*) \tag{27.}$$

Equation (24) presents the additive decomposition of stress tensor. Equation (25) is the generalized Hoke's law. Equation (26) defines the generalized associated or non-associated flow rules for plastic strain. Equation (27) presents a set of hardening/softening evolution laws. Where h_* is the plastic moduli and λ is the plastic parameter which can be determined by Karush-Kuhn-Tucker loading-unloading conditions,

$$\begin{cases} F(\sigma, q_*) \leq 0 \\ \lambda \geq 0 \\ F\lambda = 0 \end{cases} \tag{28.}$$

In this project, the Forward Euler algorithm (explicit algorithm) is investigated. In Forward Euler method, stress tensor increment can be expressed as:

$$\Delta\sigma_{mn} = E_{mnpq} \Delta\varepsilon_{pq} - E_{mnpq} \frac{{}^{cross}n_{rs} E_{rstu} \Delta\varepsilon_{tu}}{{}^{cross}n_{ab} E_{abcd} \, {}^{cross}m_{cd} - \xi_* h_*} \, {}^{cross}m_{pq} \tag{29.}$$

the plastic parameter λ:

$$\lambda = \frac{{}^{cross}n_{rs} E_{rstu} \Delta\varepsilon_{tu}}{{}^{cross}n_{ab} E_{abcd} \, {}^{cross}m_{cd} - \xi_* h_*} \tag{30.}$$

and the continuum tangent stiffness tensor:

$${}^{cont}E^{ep}_{mnpq} = E_{mnpq} - \frac{E_{mntu} \, {}^{n}m_{tu} \, {}^{n}n_{rs} E_{rspq}}{{}^{n}n_{ab} E_{abcd} \, {}^{n}m_{cd} - {}^{n}\xi_* \, {}^{n}h_*} \tag{31.}$$

where ${}^{cross}()$ denotes the starting elastic–plastic point for that increment where the combined stress and internal variable state crosses the yield surface.

$$n = \frac{\partial F}{\partial \sigma},\ \xi_* = \frac{\partial F}{\partial q_*}$$

3. Implementation and object-oriented design

To facilitate code reuse and to provide for a design which is more flexible and extensible, object-oriented design principle can be applied into template material implementation. As discussed in above sections, the main elements of an elastic-plastic material can be abstracted into separate classes. This process is so called class identification and abstraction.

1) PressureDependentElastic3D

The pressure dependent isotropic elastic material object is responsible for the elastic behavior. The major function is providing the elastic stiffness tensor E_{ijkl}.

2) YieldSurface_MD

The yield surface of Manzari & Dafalias model (1997) object is responsible for providing the value of yield function F given stress and internal variables state, as well as n_{ij} （the partial derivative of F to stress σ） and ξ_* (the partial derivative of F to internal variables q_*).

3) PotentialSurface_MD

The potential surface of Manzari & Dafalias model (1997) object is responsible for providing the value of flow direction m_{ij} (the partial derivative of Q to stress σ) given stress and internal variables state.

4) EvolutionLaw_MD

The evolution law of Manzari & Dafalias model (1997) object is responsible for providing the value of plastic moduli h_* , updating internal variables and state parameters. The model parameters of Manzari & Dafalias model (1997) are also stored in this object for convenience.

5) EPState

The elastic-plastic state object contains the elastic-plastic state of material point, such as stresses, strains, internal variables, state parameters and stiffness tensor E^{ep}. This object also provides interfaces to access those material state quantities.

6) Template3Dep

The template elastic-plastic material object is a container class for storing and access above types of objects: **PressureDependentElastic3D, YieldSurface_MD, PotentialSurface_MD, EvolutionLaw_MD, EPState.** At current implementation of OpenSees, integration algorithms are operations of **Template3Dep** object.

The class architecture of template elastic-plastic material can be described as Figure (1)

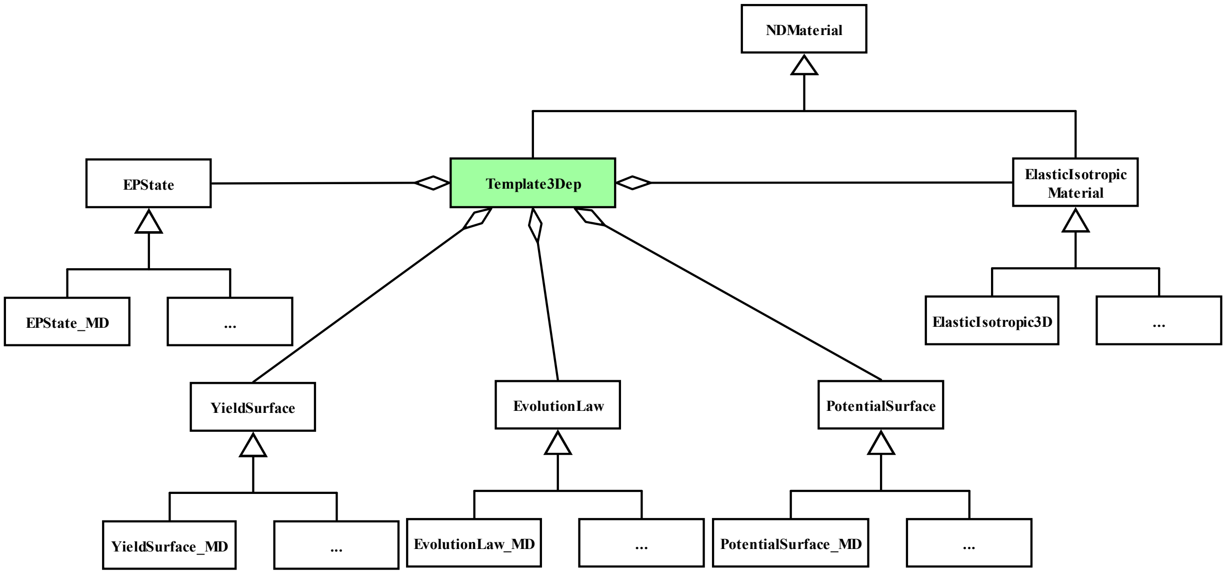

Figure 2. Class architecture

4 Discussions

In this section, some comments about the template elastic-plastic material frame which is currently implemented in OpenSees will be discussed. Part of these suggestions has been applied in this project.

1) State parameter

In Manzari & Dafalias model (1997), a state parameter ψ is introduced. In this project, the concept of state parameter is extended to the quantities which determined by state quantities and will not vary if state is not change. Usually, such quantities have following characteristic: calculated by relative complicated operations such as tensor operations, or have special physical meaning of specific material. To save time and be convenient to track, those quantities will be stored and updated when the state changes. For Manzari & Dafalias model (1997), the state parameters will include: D (dilatancy coefficient), e (void ratio), n (unite deviatoric stress-ratio tensor) and ψ.

Current implementation in OpenSees, some quantities, such as e and ψ, are stored in **EPState** object. From the philosophy of object-oriented design, it is not appropriate way to do so. The state quantities which are stored in **EPState** object should be the general quantities for elastic-plastic materials, but those state parameters is related to particular material model. An alternative way to avoid it is to treat those state parameters as the way used to treat internal variables.

2) Evolution Law

Current implementation of evolution laws in OpenSees is corresponding to the type (scalar or tensor) of internal variables. Such implementation requires developing new evolution law classes based on the number of internal variables in one specific material model. A more elegant way is introducing new evolution law classes corresponding to the new material model which means only one evolution law for one material model. Since this change will also affect some part in integration algorithm, pseudo-codes are presented as following:

```
Evolution law -> EvolutionLaw_MD
EvolutionLaw_MD::UpdateInternalVar(EPState *EPS, int whichone, double delta_lambda )
{
    switch (whichone){
              case 1: update_alpha (EPS, delta_lambda, …);
              case 2: update_m (EPS, delta_lambda, …);
              …
    }
}

Template3Dep::ForwardEulerEPState {
…
for (i = 1 to NumofInternalVar) {
    EL->UpdateInternalVar(forwardEPS, i, delta_lambda);
}
…
}
```

3) Integration Algorithm

The internal variables are updated inside the current implementation of Forward Euler method. As we discussed in the object-oriented design section, this task should be a part of evolution laws. Therefore, an appropriate way is to invoke corresponding updating operations of evolution law objects to do this. Since integration algorithms is relative independent part compared with the other parts aggregated in **Template3Dep** object, and play a role of delegation object (given strain increment, return back the updated material state). It may be better to separate integration algorithm from **Template3Dep** class, and generate new classes for integration algorithms.

EVALUATION OF MODEL PERFORMANCE

The calibration of the critical state two-surface plasticity model constants can be done on the basis of the results of conventional triaxial compression and extension tests. A list of model constants is presented in Table 1.

Table 1. Material parameters of the critical state two-surface plasticity model for Nevada sand [5]

Elastic	G_0 (kPa)	31400	Hardening	h_0	1500
	K_0 (kPa)	31400		m	0.05
	a	0.6		c_m	0.
Critical state	M_c	1.14	State parameter	k_c^b	3.975
	M_e	1.14		k_e^b	2.0
	λ	0.025		k_c^d	4.2
	$(e_c)_{ref}$	0.8		k_e^d	0.07
Dilatancy	A_0	0.64			

The Manzari-Dafalias model was already implemented in OpenSees but it was not working. The attempt of this project was trying to debug the implemented version and make it work. After finding some problems in the implementation and debugging the program in constitutive level, the results of the simulation for one increment of applied strain in constitutive level (using constitutive driver) was compared to the same increment in an excel file with the same approach and every thing was fine. Then a single element (figure 3) was considered in global finite element level and using displacement control approach the behavior of loose and dense sand sample was simulated in constant-p drained triaxial loading. Figures 3 and 4 show the results of simulation in OpenSees.

As it can be observed from the result, the hardening, dilation and contraction response can be captured pretty well with the model but the missing point is simulation of softening beyond the peak point in simulation of drained loading on dense sand (convergence problem).

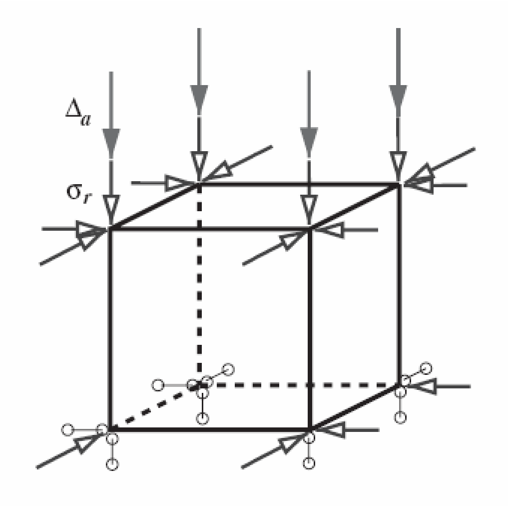

Figure 3. One element triaxial setup

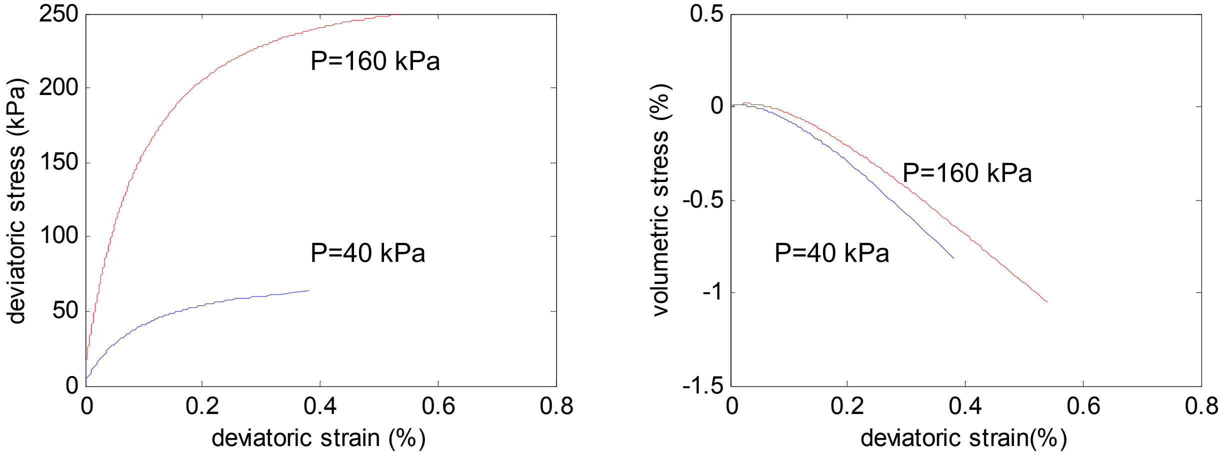

Figure 4. Model simulation of drained triaxial constant-p tests;
dense sand (e_{in}=0.65) , p=40 or 160 kPa

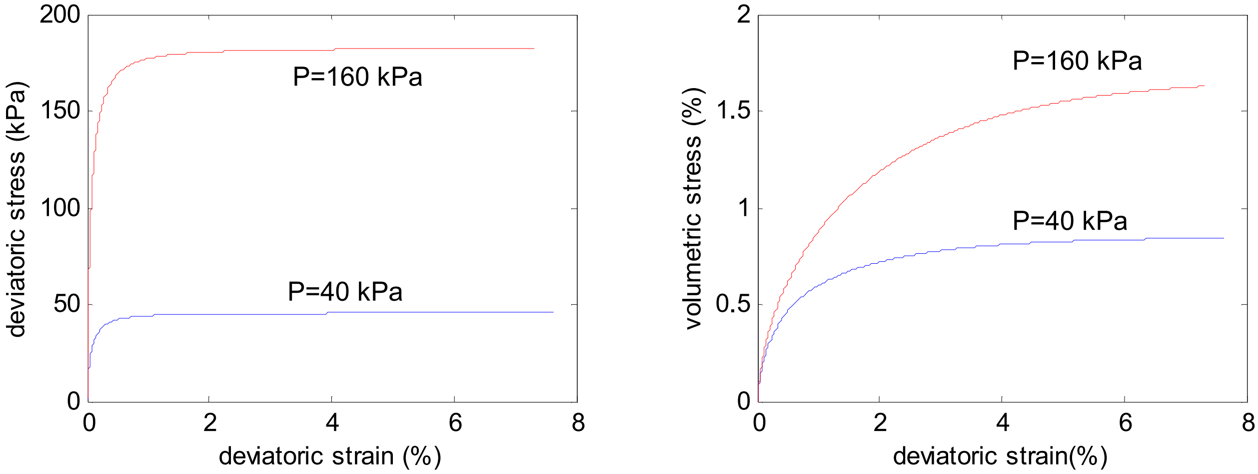

Figure 5. Model simulation of drained triaxial constant-p tests;
loose sand (e_{in}=0.87) , p=40 or 160 kPa

In the next step, using MATLAB, the Manzari-Dafalias model was examined (just in constitutive level). In this program, instead of forth and second order tensors, every thing was simplified to matrixes and vectors. Two approached was examined separately:

1- Calculating loading index and stress rate increment using equations (6.) and (7.) which were introduced in original MAnzari-Dafalias paper
2- Calculating loading index and stress rate increment using equations (29.) and (30.) which were generally used in Template3D of OpenSees

However analytical both approaches are exactly the same but these two approaches were considered just for observing why the implemented version in OpenSees is not working!

The Model simulation of three undrained triaxial tests starting at three different void ratios, are presented in figures 6 and 7. These behavior were simulated in Manzari-Dafalias paper and the results of figure 6 (using approach1) is exactly same as those reported in Manzari-Dafalias (1997) paper but there are some differences between figure 7 (using approach 2) and figure 6 (using approach 1) and in case of e=0.85, some problem occurs around 2% of axial strain! The reason is still under investigation and hopefully after solving this problem, the remaining problem in OpenSees can be found.

At this point it can just be claimed that pure explicit method needs really small increments of strain and is really sensitive for this model. Some better approaches for integration are introduced in reference [2].

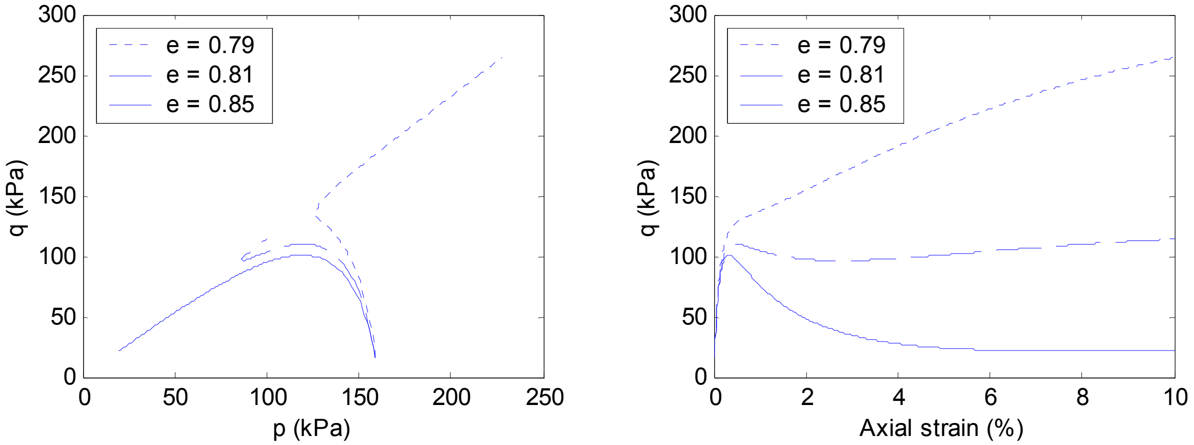

Figure 6. Model simulation of three undrained triaxial tests starting at three different void ratios, two at a state looser and one denser than critical (e_c=0.8 for p=160 kPa); results of simulation with matlab program using Manzari Dafalias approach

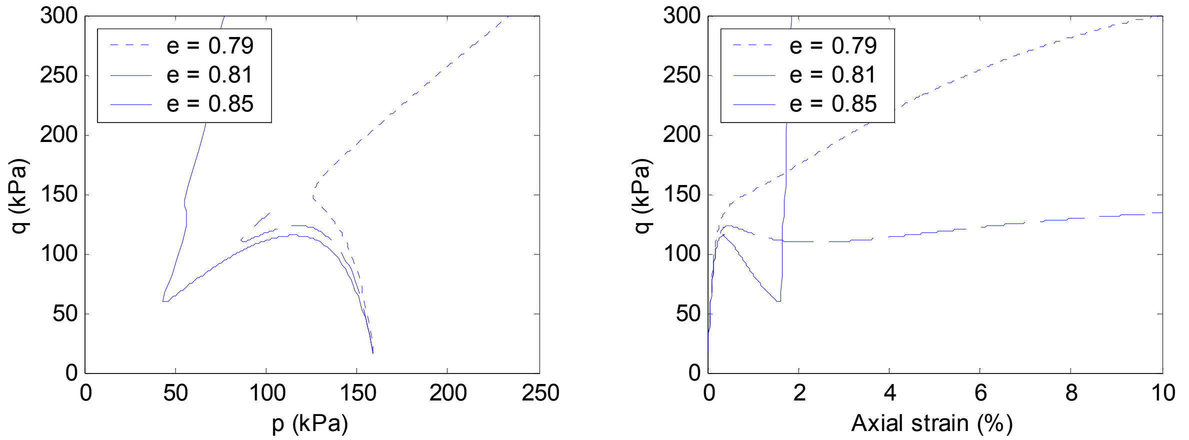

Figure 7. Model simulation of three undrained triaxial tests starting at three different void ratios, two at a state looser and one denser than critical (e_c=0.8 for p=160 kPa); results of simulation with matlab program using Template3D approach

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