104.5 Line Search Technique for Constitutive Elastic-Plastic Integration

This section is entirely based on Jeremić (2001). There exist a repetition of some previously defined equations...

104.6 Elastic and Elastic–Plastic Material Models for Solids

In this section we present elements of general elastic and elastic–plastic material models for engineering materials. We describe various forms of the yield functions, plastic flow directions and hardening and softening laws.

104.6.1 Elasticity

DSL COMMANDS for the elastic material models are given in section 205.3 on page 702.

In linear elasticity the relationship between the stress tensor σ_{ij} and the strain tensor ϵ_{kl} can be represented in the following form:

$$\sigma_{ij} = \sigma\left(\epsilon_{ij}\right) \tag{104.155}$$

If we assume the existence of a strain energy function[58] $W\left(\epsilon_{ij}\right)$ then the stress strain relation is:

$$\sigma_{ij} = \frac{\partial W\left(\epsilon_{ij}\right)}{\partial \epsilon_{ij}} \tag{104.156}$$

The introduction of the strain energy density function into elasticity is due to Green, and elastic solids for which such a function is assumed to exist are called Green elastic or hyperelastic solids.

[58] per unit volume.

Linearization of an elastic continuum is carried out with respect to a reference configuration which is stress free at temperature T_0, so that ${}^0\sigma_{ij} = 0$. If we denote as E_{ijkl} an isothermal modulus tensor, then under isothermal conditions, we obtain the generalized Hooke's law:

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl} \tag{104.157}$$

where E_{ijkl} is the fourth order elastic stiffness tensor with 81 independent components in total. The elastic stiffness tensor features both minor symmetry $E_{ijkl} = E_{jikl} = E_{ijlk}$ and major symmetry $E_{ijkl} = E_{klij}$ (Jeremić and Sture, 1997). The number of independent components for such elastic stiffness tensor is 21 (Spencer, 1980).

$$E_{ijkl} = \left|\frac{\partial^2 W}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\right|_{\epsilon=0} = \left|\frac{\partial^2 W}{\partial\epsilon_{kl}\,\partial\epsilon_{ij}}\right|_{\epsilon=0} \tag{104.158}$$

We will restrain our considerations to the isotropic case. The most general form of the isotropic tensor of rank 4 has the following representation:

$$I_4 = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} \tag{104.159}$$

If E_{ijkl} has this form then in order to satisfy the symmetry condition[59] $E_{ijkl} = E_{jikl}$ we must have $\nu = \mu$. The symmetry condition[60] $E_{ijkl} = E_{klji}$ is then automatically satisfied. The elastic constant tensor has the following form:

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{104.160}$$

where λ and μ are the Lamé coefficients:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad ; \quad \mu = \frac{E}{2(1+\nu)} \tag{104.161}$$

and E and ν are Young's Modulus and Poisson's ratio respectively. The symmetric part of the fourth order unit tensor is :

$$I^{sym}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jl}\right) \tag{104.162}$$

and can be found as multiplier of μ in equation (104.160). Equation (104.160) can be written in terms of E and ν as:

$$E_{ijkl} = \frac{E}{2(1+\nu)}\left(\frac{2\nu}{1-2\nu}\,\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{104.163}$$

The same relation in terms of bulk modulus K and shear modulus G is:

[59] symmetry in stress tensor.
[60] existence of strain energy function.

$$E_{ijkl} = K\delta_{ij}\delta_{kl} + G\left(-\frac{2}{3}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{104.164}$$

where K and G are given as:

$$K = \lambda + \frac{2}{3}\mu \quad ; \quad G = \mu \tag{104.165}$$

The relation between the strain tensor, ϵ_{kl} and the stress tensor, σ_{ij} is:

$$\epsilon_{kl} = D_{klpq}\sigma_{pq} \tag{104.166}$$

where D_{klpq} is the elastic compliance fourth order tensor, defined as:

$$D_{klpq} = \frac{-\lambda}{2\mu\,(3\lambda + 2\mu)}\;\delta_{kl}\delta_{pq} + \frac{1}{4\mu}\left(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}\right) \tag{104.167}$$

or in terms of E and ν:

$$D_{klpq} = \frac{1+\nu}{2E}\left(\frac{-2\nu}{1+\nu}\;\delta_{kl}\delta_{pq} + \delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}\right) \tag{104.168}$$

of in terms of K and G:

$$D_{klpq} = \frac{1}{9K}\;\left(\delta_{kl}\delta_{pq}\right) + \frac{1}{2G}\left(-\frac{1}{3}\delta_{kl}\delta_{pq} + \frac{1}{2}\left(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}\right)\right) \tag{104.169}$$

It is worthwhile noting that the part adjacent to the inverse of the bulk modulus K:

$$\left(\delta_{kl}\delta_{pq}\right)$$

controls the volumetric response and that the part adjacent to the inverse of the shear modulus G:

$$\left(-\frac{1}{3}\delta_{kl}\delta_{pq} + \frac{1}{2}\left(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}\right)\right)$$

controls the shear response! This note will prove useful later on. Linear transformation of the stress tensor σ_{pq} into itself, i.e. σ_{ij} is defined as:

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl} = E_{ijkl}D_{klpq}\sigma_{pq} \tag{104.170}$$

where

$$E_{ijkl}D_{klpq} = \frac{1}{2}\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) = I^{sym}_{ijpq} \tag{104.171}$$

104.6.1.1 Elastic Model

Linear elastic law is the simplest one and assumes constant Young's modulus E and constant Poisson's Ration ν.

104.6.1.2 Non–linear Elastic Model #1

This nonlinear model (Janbu, 1963), (Duncan and Chang, 1970) assumes dependence of the Young's modulus on the minor principal stress $\sigma_3 = \sigma_{min}$ in the form

$$E = Kp_a\left(\frac{\sigma_3}{p_a}\right)^n \tag{104.172}$$

Here, p_a is the atmospheric pressure in the same units as E and stress. The two material constants K and n are constant for a given void ratio.

104.6.1.3 Non–linear Elastic Model #2

If Young's modulus and Poisson's ratio are replaced by the shear modulus G and bulk modulus K the non–linear elastic relationship can be expressed in terms of the normal effective mean stress p as

$$G \quad \text{and/or} \quad K = AF(e, OCR)p^n \tag{104.173}$$

where e is the void ratio, OCR is the overconsolidation ratio and $p = \sigma_{ii}/3$ is the mean effective stress (Hardin, 1978).

104.6.1.4 Lade's Non–linear Elastic Model

Lade and Nelson (1987) and Lade (1988a) proposed a nonlinear elastic model based on Hooke's law in which Poisson ratio ν is kept constant. According to this model, Young's modulus can be expressed in terms of a power law as:

$$E = M\, p_a \left(\left(\frac{I_1}{p_a}\right)^2 + \left(6\frac{1+\nu}{1-2\nu}\right)\frac{J_{2D}}{p_a^2} \right)^{\lambda} \tag{104.174}$$

where $I_1 = \sigma_{ii}$ is the first invariant of the stress tensor and $J_{2D} = (s_{ij}s_{ij})/2$ is the second invariant of the deviatoric stress tensor $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$. The parameter p_a is atmospheric pressure expressed in the same unit as E, I_1 and $\sqrt{J_{2D}}$ and the modulus number M and the exponent λ are constant, dimensionless numbers.

104.6.1.5 Cross Anisotropic Linear Elastic Model

104.6.2 Yield Functions

The typical plastic behavior of frictional materials is influenced by both normal and shear stresses. It is usually assumed that there exists a yield surface F in the stress space that encompasses the elastic region. States of stress inside the yield surface are assumed to be elastic (linear or non–linear). Stress states on the surface are assumed to produce plastic deformations. Yield surfaces for geomaterials are usually shaped as asymmetric tar drops with smoothly rounded triangular cross sections. In addition to that, simpler yield surfaces, based on the Drucker–Prager cone or Mohr–Coulomb hexagon can also be successfully used if matched with appropriate hardening laws. Yield surface shown in Figure 104.6 Lade (1988b) represent typical meridian plane trace for an isotropic granular material. Line BC represents stress path for conventional triaxial compression test. Figure 104.7 represents the view of the yield surface traces in the deviatoric plane.

104.6.3 Plastic Flow Directions

Plastic flow directions are traditionally derived from a potential surface which to some extent reassembles the yield surface. Potential surfaces for metals are the same as their yield surfaces but experimental evidence suggests that it is not the case for geomaterials. The non–associated flow rules, used in geomechanics, rely on the potential surface, which is different from the yield surface, to provide the plastic flow directions. It should be noted that the potential surface is used for convenience and there is no physical reason to assume that the plastic strain rates are related to a potential surface Q (Vardoulakis and Sulem, 1995). Instead of defining a plastic potential, one may assume that the plastic flow direction is derived from an tensor function which does not have to possess a potential function.