701.1 Chapter Summary and Highlights

701.2 Stress and Strain

This section reviews small deformation stress and strain measures used in this report. Some of the equations, particularly for stress and strain rotations were developed with help from ChatGPT.

701.2.1 Stress

In this work, the tensile stress is assumed positive, and in general we follow classical strength of materials (mechanics of materials) conventions for stress and strain. The stress tensor σ_{ij} is defined as

$$\sigma_{ij} = \lim_{A_i \to 0} \frac{F_j}{A_i} \tag{701.1}$$

where F_j is a traction (force) in the j direction and A_i is an infinitesimal surface area with normal in i direction. Cauchy stress tensor has a total of nine components, six of which are independent (symmetry $\sigma_{ij} = \sigma_{ji}$):

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{x} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{y} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{z} \end{pmatrix} \tag{701.2}$$

In small deformation theory, this stress is symmetric, that is, $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$. There are only six independent components and sometimes the stress can be expressed in the vector form

$$\boldsymbol{\sigma} = \left\{\sigma_{xx},\ \sigma_{yy},\ \sigma_{zz}, \sigma_{xy},\ \sigma_{yz},\ \sigma_{zx}\right\} \tag{701.3}$$

The principle stresses σ_1, σ_2, and σ_3 ($\sigma_1 \geq \sigma_2 \geq \sigma_3$) are the eigenvalues of the symmetric tensor σ_{ij} in Equation 701.2 and can be obtained by solving the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma \end{vmatrix} = \mathbf{0} \tag{701.4}$$

or in alternative form

$$\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0 \tag{701.5}$$

The three first-type stress invariants are then

$$\begin{aligned} I_1 &= \sigma_{ii} \\ &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\ &= \sigma_1 + \sigma_2 + \sigma_3 \end{aligned} \tag{701.6}$$

$$\begin{aligned} I_2 &= \frac{1}{2} \sigma_{ij} \sigma_{ji} \\ &= -(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) \\ &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \end{aligned} \tag{701.7}$$

$$
\begin{aligned}
I_3 &= \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} = \det(\sigma_{ij}) \\
&= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - (\sigma_{xx}\sigma_{yz}^2 + \sigma_{yz}\sigma_{zx}^2 + \sigma_{zz}\sigma_{xy}^2) \\
&= \sigma_1\sigma_2\sigma_3
\end{aligned} \tag{701.8}
$$

The stress σ_{ij} can be decomposed into the hydrostatic stress $\sigma_m\delta_{ij}$ and deviatoric stress s_{ij} as $\sigma_{ij} = \sigma_m\delta_{ij} + s_{ij}$, with the definitions

$$
\sigma_m = \frac{1}{3}I_1, \quad s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{701.9}
$$

where δ_{ij} is the Kronecker operator such that $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

Since both hydrostatic and deviatoric stresses are stress tensors, they have their own coordinate-independent stress invariants respectively. The three invariants of the hydrostatic stress are

$$
I_1 = 3\sigma_m = I_1, \quad I_2 = -3\sigma_m^2 = -\frac{1}{3}I_1^2, \quad I_3 = \sigma_m^3 = \frac{1}{27}I_1^3 \tag{701.10}
$$

Since I_1, I_2 and I_3 are all simple functions of I_1, the hydrostatic stress state can therefore be represented by only one variable I_1.

The three eigenvalues of the deviatoric stresses s_{ij} are called principal deviatoric stresses, with the order $s_1 \geq s_2 \geq s_3$. The three invariants of the deviatoric stress are

$$
J_1 = s_{ii} = 0 \tag{701.11}
$$

$$
\begin{aligned}
J_2 &= \frac{1}{2}s_{ij}s_{ji} \\
&= \frac{1}{3}I_1^2 + I_2 \\
&= \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\
&= -(s_{xx}s_{yy} + s_{yy}s_{zz} + s_{zz}s_{xx}) + (s_{xy}^2 + s_{yz}^2 + s_{zx}^2) \\
&= \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) = -(s_1s_2 + s_2s_3 + s_3s_1)
\end{aligned} \tag{701.12}
$$

$$
\begin{aligned}
J_3 &= \frac{1}{3}s_{ij}s_{jk}s_{ki} = \det(s_{ij}) \\
&= I_3 + \frac{1}{3}I_1I_2 + \frac{2}{27}I_1^3 = I_3 - \frac{1}{3}I_1J_2 - \frac{1}{27}I_1^3 \\
&= \frac{1}{27}(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2) \\
&= s_{xx}s_{yy}s_{zz} + 2s_{xy}s_{yz}s_{zx} - (s_{xx}s_{yz}^2 + s_{yy}s_{zx}^2 + s_{zz}s_{xy}^2) \\
&= s_1s_2s_3
\end{aligned} \tag{701.13}
$$

The deviatoric stress state can therefore be represented by only two variables J_2 and J_3.

Combining hydrostatic and deviatoric stress, we can conclude that the stress state can be be represented by three variables I_1, J_2 and J_3. Using the three invariants (I_1, J_2, J_3) or its equivalents instead of the nine components of σ_{ij} is widely used in geomechanics.

The stress state may also be described in three dimensional space (p, q, θ_σ), defined as

$$p = -\frac{1}{3} I_1 \tag{701.14}$$

$$q = \sqrt{3J_2} \tag{701.15}$$

$$\theta_\sigma = \frac{1}{3} \arccos \left(\frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}} \right) \tag{701.16}$$

where $\theta_{\sigma_{ij}}$ is the stress Lode's angle $(0 \leq \theta_{\sigma_{ij}} \leq \pi/3)$. A stress state with $\theta_\sigma = 0$ corresponds to the meridian of conventional triaxial compression (CTC), while $\theta_\sigma = \pi/3$ to the meridian of conventional triaxial extension (CTE). The relationship between $(\sigma_1, \sigma_2, \sigma_3)$ and (p, q, θ_σ) is

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = -p + \frac{2}{3} q \begin{pmatrix} \cos \theta_\sigma \\ \cos(\theta_\sigma - \frac{2}{3}\pi) \\ \cos(\theta_\sigma + \frac{2}{3}\pi) \end{pmatrix} \tag{701.17}$$

The line of the principal stress space diagonal is called hydrostatic axis. Any plane perpendicular to the hydrostatic axis is an deviatoric plane, or π plane. The Haigh-Westergaard three dimensional stress coordinate system $(\xi, \rho, \theta_\sigma)$ Chen and Han (1988a), is defined as

$$\xi = \frac{I_1}{\sqrt{3}} = -\sqrt{3} p \tag{701.18}$$

$$\rho = \sqrt{2J_2} = \sqrt{\frac{2}{3}} q \tag{701.19}$$

The Haigh-Westergaard invariants have physical meanings. ξ is the distance of the deviatoric plane to the origin of the Haigh-Westergaard coordinates, and ρ is the distance of a stress point to the hydrostatic line and represents the magnitude of the deviatoric stress. The projections of the axes σ_1, σ_2 and σ_3 on the deviatoric plane are assumed σ'_1, σ'_2 and σ'_3 respectively. (ρ, θ_σ) is the polar coordinate system in the deviatoric plane with the σ'_1 the polar axis and θ_σ the polar angle. The relationship between $(\sigma_1, \sigma_2, \sigma_3)$ and $(\xi, \rho, \theta_\sigma)$ is

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \frac{1}{\sqrt{3}} \xi + \sqrt{\frac{2}{3}} \rho \begin{pmatrix} \cos \theta_\sigma \\ \cos(\theta_\sigma - \frac{2}{3}\pi) \\ \cos(\theta_\sigma + \frac{2}{3}\pi) \end{pmatrix} \tag{701.20}$$

701.2.1.1 Stress Rotations

Two-dimensional stress rotation. In two dimensions, the Cauchy stress tensor σ_{ij} $(i, j = 1, 2)$ transforms under a change of orthonormal basis according to the second-order tensor transformation rule.

$$\sigma'_{ij} = Q_{ik} \, Q_{jl} \, \sigma_{kl} \tag{701.21}$$

The rotation tensor Q_{ij} is orthogonal and satisfies the following properties.

$$Q_{ik} Q_{jk} = \delta_{ij}, \qquad \det Q_{ij} = +1 \tag{701.22}$$

For a planar rotation by angle θ, the rotation tensor may be written explicitly.

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{701.23}$$

The transformed stress components σ'_{ij} follow directly from the tensor transformation law.
In details:

Three-dimensional stress rotation. In three dimensions, the Cauchy stress tensor σ_{ij} $(i, j = 1, 2, 3)$ transforms under a general rigid-body rotation according to the same second-order tensor transformation rule.

$$\sigma'_{ij} = Q_{ik}\, Q_{jl}\, \sigma_{kl} \tag{701.24}$$

The rotation tensor Q_{ij} defines a proper orthogonal transformation of the coordinate basis and satisfies the orthogonality conditions.

$$Q_{ik}Q_{jk} = \delta_{ij}, \qquad Q_{ki}Q_{kj} = \delta_{ij} \tag{701.25}$$

The traction vector t_i acting on a plane with unit normal n_j is defined as follows.

$$t_i = \sigma_{ij} n_j \tag{701.26}$$

The normal and shear components of the stress acting on the plane are obtained by projection.

$$\sigma_n = n_i \sigma_{ij} n_j \tag{701.27}$$

$$\tau_i = t_i - \sigma_n n_i \tag{701.28}$$

The principal stresses and principal directions are obtained from the eigenvalue problem.

$$\sigma_{ij} v_j^{(a)} = \sigma^{(a)} v_i^{(a)}, \qquad a = 1, 2, 3 \tag{701.29}$$

Stress invariants are preserved under rigid-body rotations. The first invariant and deviatoric stress tensor are given by

$$I_1 = \sigma_{ii} \tag{701.30}$$

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{701.31}$$

701.2.2 Strain

Point $P(x_i)$ and nearby point $Q(x_i + \mathrm{d}x_i)$ displace due to applied loading to new positions $P(x_i + U_i)$ and $Q(u_i + (\partial u_i/\partial x_j)\mathrm{d}x_j)$. We can define a displacement gradient tensor $u_{i,j}$ as

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} \tag{701.32}$$

Matrix form of the displacement gradient can decomposed into the symmetric and antisymmetric parts

$$\begin{aligned}\begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix} &= \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ \frac{1}{2}(u_{2,1}+u_{1,2}) & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ \frac{1}{2}(u_{3,1}+u_{1,3}) & \frac{1}{2}(u_{3,2}+u_{2,3}) & u_{3,3} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \frac{1}{2}(u_{1,2}-u_{2,1}) & \frac{1}{2}(u_{1,3}-u_{3,1}) \\ \frac{1}{2}(u_{2,1}-u_{1,2}) & 0 & \frac{1}{2}(u_{2,3}-u_{3,2}) \\ \frac{1}{2}(u_{3,1}-u_{1,3}) & \frac{1}{2}(u_{3,2}-u_{2,3}) & 0 \end{pmatrix}\end{aligned} \tag{701.33}$$

or

$$u_{i,j} = \epsilon_{ij} + w_{ij} \tag{701.34}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{701.35}$$

$$w_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{701.36}$$

The symmetric part of the deformation gradient tensor, ϵ_{ij}, is the small deformation strain tensor [1], while the antisymmetric part of the deformation gradient tensor, w_{ij}, is the rotation motion tensor. The matrix form of the strain ϵ_{ij} is

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} = \begin{pmatrix} \epsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_{z} \end{pmatrix} \tag{701.37}$$

The engineering strain is usually expressed in the vector form

$$\boldsymbol{\epsilon} = \left\{\epsilon_x,\ \epsilon_y,\ \epsilon_z, \gamma_{xy},\ \gamma_{yz},\ \gamma_{zx}\right\}^T \tag{701.38}$$

Note that the engineering shear strain γ_{ij} is the double of the corresponding strain component ϵ_{ij}.

Similar to the stress tensor, the strain tensor also has three principle strains $\epsilon_i(\epsilon_1 \geq \epsilon_2 \geq \epsilon_3)$, and three strain invariants I'_1, I'_2, and I'_3, defined as

$$\begin{aligned} I'_1 &= \epsilon_{ii} \\ &= \epsilon_v \\ &= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\ &= \epsilon_1 + \epsilon_2 + \epsilon_3 \end{aligned} \tag{701.39}$$

[1] Here the second and higher order derivative terms are neglected due to the small deformation assumption.

$$\begin{aligned} I_2' &= \frac{1}{2}\epsilon_{ij}\epsilon_{ji} \\ &= -(\epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx}) + (\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{zx}^2) \\ &= -(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1) \end{aligned} \tag{701.40}$$

$$\begin{aligned} I_3' &= \frac{1}{3}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki} = \det(\epsilon_{ij}) \\ &= \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} + 2\epsilon_{xy}\epsilon_{yz}\epsilon_{zx} - (\epsilon_{xx}\epsilon_{yz}^2 + \epsilon_{yz}\epsilon_{zx}^2 + \epsilon_{zz}\epsilon_{xy}^2) \\ &= \epsilon_1\epsilon_2\epsilon_3 \end{aligned} \tag{701.41}$$

The first strain invariant is also called the volumetric strain ϵ_v.

The strain ϵ_{ij} can be decomposed into the hydrostatic strain $\epsilon_m\delta_{ij}$ and deviatoric strain e_{ij} through $\epsilon_{ij} = \epsilon_m\delta_{ij} + e_{ij}$ where:

$$\epsilon_m = \frac{1}{3}I_1', \quad e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij} \tag{701.42}$$

Since both hydrostatic and deviatoric strains are strain tensors, they have their own strain invariants respectively. The three invariants of the hydrostatic strain are

$$I_1' = 3\epsilon_m = I_1', \quad I_2' = -3\epsilon_m^2 = -\frac{1}{3}(I_1')^2, \quad I_3' = \epsilon_m^3 = \frac{1}{27}(I_1')^3 \tag{701.43}$$

The hydrostatic strain state can therefore be represented by only one variable I_1'.

The three eigenvalues of the deviatoric strains e_{ij} are called principal deviatoric strains, with the order $e_1 \geq e_2 \geq e_3$. The three invariants of the deviatoric strain are

$$J_1' = e_{ii} = 0 \tag{701.44}$$

$$\begin{aligned} J_2' &= \frac{1}{2}e_{ij}e_{ji} \\ &= \frac{1}{3}(I_1')^2 + I_2' \\ &= \frac{1}{6}[(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2] \\ &= -(e_{xx}e_{yy} + e_{yy}e_{zz} + e_{zz}e_{xx}) + (e_{xy}^2 + e_{yz}^2 + e_{zx}^2) \\ &= \frac{1}{2}(e_1^2 + e_2^2 + e_3^2) = -(e_1e_2 + e_2e_3 + e_3e_1) \end{aligned} \tag{701.45}$$

$$\begin{aligned} J_3' &= \frac{1}{3}e_{ij}e_{jk}e_{ki} = \det(e_{ij}) \\ &= I_3' + \frac{1}{3}I_1'I_2' + \frac{2}{27}(I_1')^3 = I_3 - \frac{1}{3}I_1'J_2' - \frac{1}{27}(I_1')^3 \\ &= \frac{1}{27}(2\epsilon_1 - \epsilon_2 - \epsilon_3)(2\epsilon_2 - \epsilon_3 - \epsilon_1)(2\epsilon_3 - \epsilon_1 - \epsilon_2) \\ &= e_{xx}e_{yy}e_{zz} + 2e_{xy}e_{yz}e_{zx} - (e_{xx}e_{yz}^2 + e_{yy}s_{zx}^2 + e_{zz}e_{xy}^2) \\ &= e_1e_2e_3 \end{aligned} \tag{701.46}$$

The deviatoric strain state can therefore be represented by only two variables J_2' and J_3'.

Combining the hydrostatic and deviatoric strain, we can conclude that the strain state can be be represented by three variables I_1', J_2' and J_3'.

Strain state may also be represented with another three invariant $(\epsilon_p, \epsilon_q, \theta_\epsilon)$, defined as

$$\epsilon_p = -I_1' = -\epsilon_v \tag{701.47}$$

$$\epsilon_q = 2\sqrt{\frac{J_2'}{3}} \tag{701.48}$$

$$\theta_\epsilon = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3'}{\sqrt{(J_2')^3}}\right) \tag{701.49}$$

where θ_ϵ is the strain Lode's angle and $0 \leq \theta_\epsilon \leq \pi/3$. The relationship between $(\epsilon_1, \epsilon_2, \epsilon_3)$ and $(\epsilon_p, \epsilon_q, \theta_\epsilon)$ is

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = -\frac{1}{3}\epsilon_p + \sqrt{\frac{3}{2}}\epsilon_q \begin{pmatrix} \cos\theta_\epsilon \\ \cos(\theta_\epsilon - \frac{2}{3}\pi) \\ \cos(\theta_\epsilon + \frac{2}{3}\pi) \end{pmatrix} \tag{701.50}$$

701.2.2.1 Strain Rotations

Two-dimensional strain rotation. In two dimensions, the infinitesimal strain tensor ε_{ij} $(i, j = 1, 2)$ transforms under a change of orthonormal basis according to the second-order tensor transformation rule.

$$\varepsilon'_{ij} = Q_{ik}\, Q_{jl}\, \varepsilon_{kl} \tag{701.51}$$

The rotation tensor Q_{ij} is orthogonal and satisfies the following properties.

$$Q_{ik}Q_{jk} = \delta_{ij}, \qquad \det Q_{ij} = +1 \tag{701.52}$$

For a planar rotation by angle θ, the rotation tensor may be written explicitly.

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{701.53}$$

The transformed strain components ε'_{ij} follow directly from the tensor transformation law.

Three-dimensional strain rotation. In three dimensions, the infinitesimal strain tensor ε_{ij} $(i, j = 1, 2, 3)$ transforms under a general rigid-body rotation according to the same second-order tensor transformation rule.

$$\varepsilon'_{ij} = Q_{ik}\, Q_{jl}\, \varepsilon_{kl} \tag{701.54}$$

The rotation tensor Q_{ij} defines a proper orthogonal transformation of the coordinate basis and satisfies the orthogonality conditions.

$$Q_{ik}Q_{jk} = \delta_{ij}, \qquad Q_{ki}Q_{kj} = \delta_{ij} \tag{701.55}$$

The infinitesimal strain tensor is defined in terms of the displacement field u_i as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{701.56}$$

where $(\cdot)_{,j}$ denotes partial differentiation with respect to the spatial coordinate x_j.
The volumetric strain is defined as the trace of the strain tensor.

$$\varepsilon_v = \varepsilon_{ii} \tag{701.57}$$

The deviatoric strain tensor is defined by removing the volumetric contribution.

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{701.58}$$

Strain invariants are preserved under rigid-body rotations, ensuring objectivity of the kinematic description.

701.2.3 Application of 2D Stress and Strain Transformation to Strain–Stress Rosettes

Strain rosette: purpose and assumptions. A strain rosette is used to determine the complete in-plane strain state at a point when the principal directions are unknown. The method relies directly on the two-dimensional strain transformation equations.

The following assumptions apply: small strains, linear elastic material behavior, isotropy, plane stress or plane strain conditions, and perfect bonding between the strain gages and the material surface.

Strain measured by a single gage. A strain gage oriented at an angle θ with respect to the x-axis measures the normal strain along its axis. This measured strain is obtained by projecting the strain tensor onto the gage direction.

$$\varepsilon_\theta = n_i n_j \varepsilon_{ij} \tag{701.59}$$

The unit vector along the gage direction is defined as

$$n_i = (\cos\theta,\ \sin\theta) \tag{701.60}$$

Expanding the projection explicitly yields

$$\varepsilon_\theta = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + 2\varepsilon_{xy}\sin\theta\cos\theta \tag{701.61}$$

This expression is identical to the two-dimensional strain transformation equation.

General three-element strain rosette. Consider a strain rosette consisting of three gages oriented at angles θ_1, θ_2, and θ_3. The measured strains are denoted by

$$\varepsilon^{(a)} = \varepsilon_{\theta_a}, \qquad a = 1, 2, 3 \tag{701.62}$$

Each measurement satisfies the transformation relation

$$\varepsilon^{(a)} = \varepsilon_{xx} \cos^2 \theta_a + \varepsilon_{yy} \sin^2 \theta_a + 2\varepsilon_{xy} \sin \theta_a \cos \theta_a \tag{701.63}$$

The three measurements form a linear system that may be solved for the unknown strain components ε_{xx}, ε_{yy}, and ε_{xy}.

Rectangular strain rosette ($0°$–$45°$–$90°$). For a rectangular rosette, the gage orientations are

$$\theta_1 = 0^\circ, \qquad \theta_2 = 45^\circ, \qquad \theta_3 = 90^\circ \tag{701.64}$$

The measured strains are denoted by ε_0, ε_{45}, and ε_{90}. Solving the transformation equations yields the in-plane strain components

$$\varepsilon_{xx} = \varepsilon_0 \tag{701.65}$$

$$\varepsilon_{yy} = \varepsilon_{90} \tag{701.66}$$

$$\varepsilon_{xy} = \varepsilon_{45} - \tfrac{1}{2} (\varepsilon_0 + \varepsilon_{90}) \tag{701.67}$$

These expressions are exact within the stated assumptions.

Principal strains and principal directions. The principal strains are obtained from the eigenvalue problem associated with the strain tensor. In two dimensions, they are given explicitly by

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \varepsilon_{xy}^2} \tag{701.68}$$

The orientation of the principal strain directions satisfies

$$\tan 2\theta_p = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{701.69}$$

Stress recovery from strain (plane stress). Assuming isotropic linear elasticity under plane stress conditions, the stress components are obtained from the strain components as

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \tag{701.70}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \tag{701.71}$$

$$\sigma_{xy} = 2G\varepsilon_{xy} = \frac{E}{1+\nu}\varepsilon_{xy} \tag{701.72}$$

The recovered stresses are expressed in the original x–y coordinate system.

Principal stresses obtained from rosette data. Once the in-plane stress components are known, the principal stresses are obtained from the stress transformation equations as

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \tag{701.73}$$

The orientation of the principal stress directions is given by

$$\tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{701.74}$$

701.3 Derivatives of Stress Invariants

In this part of the Appendix, we shall derive some useful formulae, that are rarely found[2] in texts treating elasto–plastic problems in mechanics of solid continua.
First derivative of I_1 with respect to stress tensor σ_{ij}:

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \frac{\partial \sigma_{kk}}{\partial \sigma_{ij}} = \delta_{ij}$$

First derivative of J_{2D} with respect to stress tensor σ_{ij}:

$$\frac{\partial J_{2D}}{\partial \sigma_{ij}} = \frac{\partial(\frac{1}{2}s_{mn}s_{nm})}{\partial \sigma_{ij}} = \frac{1}{2}\frac{\partial s_{mn}}{\partial \sigma_{ij}}s_{nm} + \frac{1}{2}\frac{\partial s_{nm}}{\partial \sigma_{ij}}s_{mn} =$$

$$= \frac{\partial s_{nm}}{\partial \sigma_{ij}}s_{mn} = \frac{\partial(\sigma_{nm} - \frac{1}{3}\sigma_{kk}\delta_{nm})}{\partial \sigma_{ij}}s_{mn} = (\delta_{ni}\delta_{jm} - \frac{1}{3}\delta_{nm}\delta_{ki}\delta_{jk})s_{mn} =$$

$$= (\delta_{ni}\delta_{jm} - \frac{1}{3}\delta_{nm}\delta_{ij})s_{mn} = \delta_{ni}\delta_{jm}s_{nm} - \frac{1}{3}\delta_{nm}\delta_{ij}s_{mn} = s_{ij}{}^{\dagger}$$

[2] if found at all.
†because $\delta_{nm}\delta_{ij}s_{mn} \equiv 0$

First derivative of J_{3D} with respect to stress tensor σ_{pq}:

$$\frac{\partial J_{3D}}{\partial \sigma_{pq}} = \frac{\partial(\frac{1}{3}s_{ij}s_{jk}s_{ki})}{\partial \sigma_{pq}} = \frac{1}{3}\frac{\partial s_{ij}}{\partial \sigma_{pq}}s_{jk}s_{ki} + \frac{1}{3}\frac{\partial s_{jk}}{\partial \sigma_{pq}}s_{ij}s_{ki} + \frac{1}{3}\frac{\partial s_{ki}}{\partial \sigma_{pq}}s_{ij}s_{jk} =$$
$$= \frac{\partial s_{ij}}{\partial \sigma_{pq}}s_{jk}s_{ki} = \frac{\partial(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij})}{\partial \sigma_{pq}}s_{jk}s_{ki} = (\delta_{ip}\delta_{qj} - \frac{1}{3}\delta_{ij}\delta_{kp}\delta_{qk})s_{jk}s_{ki} =$$
$$= \delta_{ip}\delta_{qj}s_{jk}s_{ki} - \frac{1}{3}\delta_{ij}\delta_{kp}\delta_{qk}s_{jk}s_{ki} = s_{qk}s_{kp} - \frac{2}{3}\delta_{pq}J_{2D} = t_{pq}{}^{\ddagger}$$

First derivative of s_{pq} with respect to stress tensor σ_{mn}, or second derivative of J_{2D} with respect to stress tensors σ_{pq} and σ_{mn}:

$$\frac{\partial s_{pq}}{\partial \sigma_{mn}} = \frac{\partial\left(\sigma_{pq} - \frac{1}{3}\delta_{pq}\sigma_{kk}\right)}{\partial \sigma_{mn}} = \frac{\partial\left(\left(\delta_{mp}\delta_{nq} - \frac{1}{3}\delta_{pq}\delta_{mn}\right)\sigma_{mn}\right)}{\partial \sigma_{mn}} =$$
$$= \left(\delta_{mp}\delta_{nq} - \frac{1}{3}\delta_{pq}\delta_{mn}\right) = p_{pqmn}$$

Second derivative of J_{3D} with respect to stress tensors σ_{pq} and σ_{mn}:

$$\frac{\partial t_{pq}}{\partial \sigma_{mn}} = \frac{\partial\left(s_{qk}s_{kp} - \frac{2}{3}\delta_{pq}J_{2D}\right)}{\partial \sigma_{mn}} = \frac{\partial\left(s_{qk}s_{kp}\right)}{\partial \sigma_{mn}} - \frac{\partial\left(\frac{2}{3}\delta_{pq}J_{2D}\right)}{\partial \sigma_{mn}} =$$
$$= \frac{\partial\left(s_{qk}s_{kp}\right)}{\partial \sigma_{mn}} - \frac{2}{3}\delta_{pq}\frac{\partial J_{2D}}{\partial \sigma_{mn}} = \frac{\partial s_{qk}}{\partial \sigma_{mn}}s_{kp} + s_{qk}\frac{\partial s_{kp}}{\partial \sigma_{mn}} - \frac{2}{3}\delta_{pq}s_{mn} =$$
$$= \left(\delta_{qm}\delta_{nk} - \frac{1}{3}\delta_{qk}\delta_{nm}\right)s_{kp} + s_{qk}\left(\delta_{km}\delta_{np} - \frac{1}{3}\delta_{kp}\delta_{nm}\right) - \frac{2}{3}\delta_{pq}s_{mn} =$$
$$= \left(\delta_{qm}s_{np} - \frac{1}{3}s_{qp}\delta_{nm}\right) + \left(s_{qm}\delta_{np} - \frac{1}{3}s_{qp}\delta_{nm}\right) - \frac{2}{3}\delta_{pq}s_{mn} =$$
$$= s_{np}\delta_{qm} + s_{qm}\delta_{np} - \frac{2}{3}s_{qp}\delta_{nm} - \frac{2}{3}\delta_{pq}s_{mn} = w_{pqmn}$$

Multiplying stiffness tensor E_{ijkl} with compliance tensor D_{klpq}:

‡since $\frac{1}{3}\delta_{ij}\delta_{kp}\delta_{qk}s_{jk}s_{ki} = \frac{1}{3}\delta_{kp}\delta_{qk}s_{ik}s_{ki} = \frac{1}{3}\delta_{qp}s_{ik}s_{ki} = \frac{2}{3}\delta_{pq}J_{2D}$ see also Chen and Han (1988a) page 222

$$
\begin{aligned}
E_{ijkl} D_{klpq} &= \\
\frac{E}{2(1+\nu)} \frac{1+\nu}{2E} \left(\frac{2\nu}{1-2\nu} \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \left(\frac{-2\nu}{1+\nu} \delta_{kl}\delta_{pq} + \delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp} \right) &= \\
\frac{1}{4} \left(\delta_{ik}\delta_{jl}\delta_{kp}\delta_{lq} + \delta_{il}\delta_{jk}\delta_{kp}\delta_{lq} + \delta_{ik}\delta_{jl}\delta_{kq}\delta_{lp} + \delta_{il}\delta_{jk}\delta_{kq}\delta_{lp} \right) &+ \\
+\frac{\nu}{2(1-2\nu)} \left(\delta_{ij}\delta_{kl}\delta_{kp}\delta_{lq} + \delta_{ij}\delta_{kl}\delta_{kq}\delta_{lp} \right) - \frac{\nu}{2(1+\nu)} \left(\delta_{ik}\delta_{jl}\delta_{kl}\delta_{pq} + \delta_{il}\delta_{jk}\delta_{kl}\delta_{pq} \right) &- \\
-\frac{\nu^2}{(1-2\nu)(1+\nu)} \delta_{ij}\delta_{kl}\delta_{kl}\delta_{pq} &= \\
\frac{1}{2} \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \right) &+ \\
+\frac{\nu}{2(1-2\nu)} \left(\delta_{ij}\delta_{kq}\delta_{kp} + \delta_{ij}\delta_{kp}\delta_{kq} \right) - \frac{\nu}{2(1+\nu)} \left(\delta_{il}\delta_{jl}\delta_{pq} + \delta_{il}\delta_{jl}\delta_{pq} \right) &- \\
-\frac{3\nu^2}{(1-2\nu)(1+\nu)} \delta_{ij}\delta_{pq} &= \\
\frac{1}{2} \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \right) &+ \\
+\frac{\nu}{2(1-2\nu)} \left(\delta_{ij}\delta_{pq} + \delta_{ij}\delta_{pq} \right) - \frac{\nu}{2(1+\nu)} \left(\delta_{ij}\delta_{pq} + \delta_{ij}\delta_{pq} \right) &- \\
-\frac{3\nu^2}{(1-2\nu)(1+\nu)} \delta_{ij}\delta_{pq} &= \\
\frac{1}{2} \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \right) &+ \\
+\frac{\nu}{(1-2\nu)} \delta_{ij}\delta_{pq} - \frac{\nu}{2(1+\nu)} \delta_{ij}\delta_{pq} &- \\
-\frac{3\nu^2}{(1-2\nu)(1+\nu)} \delta_{ij}\delta_{pq} &= \\
\frac{1}{2} \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \right) &+ \\
+\frac{\nu(1+\nu) - \nu(1-2\nu) - 3\nu^2}{(1-2\nu)(1+\nu)} \delta_{ij}\delta_{pq} &= \\
\frac{1}{2} \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \right) &= I^{sym}_{ijpq}
\end{aligned}
$$