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FINITE DYNAMIC MODEL FOR INFINITE MEDIA

By John Lysmer, M. ASCE, and Roger L. Kuhlemeyer, A. M. ASCE

INTRODUCTION

The rational approach to analytical studies of sonic flaw detection methods, blast effects, foundation vibrations, and numerous other dynamic problems is to consider them as wave propagation problems in an infinite solid. This approach has two major advantages: (1) The radiation of energy from the excited zone to the far field is properly accounted for; and (2) the concept of infinity is mathematically convenient when the aim is to find closed form analytical solutions for simple homogeneous systems.

For most of the complicated geometrics encountered in practice it is not possible to find closed form solutions and, therefore, it is necessary to resort to numerical methods of the finite difference or discrete mass type. With these methods only a finite number of nodal points can be considered; thus the numerical methods are not directly applicable to infinite systems. A general method through which an infinite system may be approximated by a finite system with a special viscous boundary condition is described herein. The resulting model may be analyzed by ordinary numerical methods but is handled with particular ease by the finite element method.

Examples utilizing the finite model for analysis of steady state foundation vibration problems are presented herein. The results clearly demonstrate: (1) The near equivalence between an infinite half space and the proposed finite model; and (2) the power of the method to analyze previously unsolvable problems.

Note.—Discussion open until January 1, 1970. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 95, No. EM4, August, 1969. Manuscript was submitted for review for possible publication on August 1, 1968.

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INFINITE MEDIA

one used by Ewing, Jardetzky, and Press (2) for the reflection and refraction of elastic waves at an interface between two media.

Incident P-Wave.—The situation for the case of an incident P-wave is shown in Fig. 2. The x-axis represents the viscous boundary and the elastic medium is located in the lower half plane. The incident wave generates two reflected waves, an S-wave which travels with the velocity

$$V_S = \sqrt{\frac{C}{\rho}} \tag{3}$$

and, a P-wave which propagates with the velocity

$$V_P = \frac{1}{s} V_S \tag{4}$$

In the previous equations $C$ is the shear modulus and $s$ is an elastic constant defined by

$$s^2 = \frac{1 - 2 \mu}{2(1 - \mu)} \tag{5}$$

in which $\mu$ is Poisson's ratio.

![Diagram of incident P-wave at viscous boundary](image)

FIG. 2.—INCIDENT P-WAVE AT VISCOUS BOUNDARY

The directions of the incident and reflected waves are related through Snell's Law

$$\cos \nu = s \cdot \cos \theta \tag{6}$$

and the velocity of the wave front along the x-axis is

$$c = V_P \cdot \sec \theta \tag{7}$$

The horizontal and vertical displacements may be written in terms of displacement potentials $\phi$ and $\psi$ as

---

3Numerals in parentheses refer to corresponding items in the Appendix I.— References.
\[ u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \]  
\[ w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \]  

in which \( \phi = \phi(x,z,t) \) and \( \psi = \psi(x,z,t) \) represent the displacements due to P-waves and S-waves, respectively. Since the wave fronts of the three waves in Fig. 2 must travel along the x-axis at same velocity \( c \), it can be shown that the displacement potentials for harmonic waves of frequency \( \omega \) must have the form

\[ \phi = \exp \left[ ik(c t + z \tan \theta - x) \right] + A \exp \left[ ik(c t - z \tan \theta - x) \right] \]  
incident P-wave

and \( \psi = B \exp \left[ ik(c t - z \tan \nu - x) \right] \)  
reflected P-wave

in which \( k \) is the wave number defined by

\[ k = \frac{\omega}{c} \]  
and \( A \) and \( B \) are the unknown amplitudes of the reflected waves. These amplitudes may be determined by the stress boundary conditions defined in Eqs. 1 and 2 which may be expressed in terms of displacement potentials as follows:

\[ 1 - 2 \frac{s^2}{v^2} \nabla^2 \phi + 2 \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x \partial z} \right) = \frac{1}{S V_S} \left( \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \right) \]  

Substitution of Eqs. 10 and 11 into Eqs. 13 and 14 yields the two linear equations

\[ (1 - 2 s^2 \cos^2 \theta - a \sin \theta) A + (\sin 2 \nu + a \cos \theta) B \]  
\[ = 2 s^2 \cos^2 \theta - 1 + a \sin \theta \]  
\[ (b \cos \nu + s^2 \sin 2 \theta) A + (\cos 2 \nu - b \sin \nu) B \]  
\[ = s^2 \sin 2 \nu - b \cos \nu \]

from which the amplitudes \( A \) and \( B \) may be determined as functions of the incident angle \( \theta \).

Incident S-Wave.—The case of an incident S-wave shown in Fig. 3 may be solved in a similar fashion. Eqs. 6 to 9 and Eqs. 12 to 14 are valid for this case but Eqs. 10 and 11 must be replaced by

\[ \phi = A \exp \left[ ik(c t - z \tan \theta - x) \right] \]  
reflected P-wave

\[ \psi = \exp \left[ ik(c t + z \tan \nu - x) \right] + B \exp \left[ ik(c t - z \tan \nu - x) \right] \]

incident S-wave reflected S-wave

Thus the linear equations applicable to define \( A \) and \( B \) for the case of an incident S-wave are as follows:

\[ (s^2 \sin 2 \theta + b \cos \nu) A + (\cos 2 \nu - b \sin \nu) B = - \cos 2 \nu - b \sin \nu \]  
\[ (-\cos 2 \nu + a \sin \theta) A + (\sin 2 \nu + a \cos \theta) B = \sin 2 \nu - a \cos \theta \]

A special case occurs when the incident angle \( \nu \) becomes smaller than the critical angle \( \nu_c \) defined by
amplitude is unity. Consequently the incident energy is

$$E_i = \frac{1}{2} \rho \, V_s \, \omega^2 \, \sin \theta$$  \hspace{1cm} (26)$$

Similarly, the reflected energy is

$$E_r = \frac{1}{2} \rho \, V_s \, \omega^2 \, A^2 \, \sin \theta + \frac{1}{2} \rho \, V_s \, \omega^2 \, B^2 \, \sin \theta$$  \hspace{1cm} (27)$$

and the energy ratio becomes

$$\frac{E_r}{E_i} = A^2 + B^2 \frac{\sin \theta}{\sin \theta}$$  \hspace{1cm} (28)$$

For a given choice of $a$ and $b$ the energy ratio depends only on the incident angle $\theta$ and Poisson's ratio $\nu$. The variation of the energy ratio with $\theta$ for Poisson's ratio equal to 0.25 is shown in Fig. 5. Other values of Poisson's ratio give similar results. A unit energy ratio corresponds to perfect reflec-

A good measure for this ability is the energy ratio defined as the ratio between the transmitted energy of the reflected waves and the transmitted energy of the incident wave. This ratio can be computed from the wave amplitudes $A$ and $B$ by considering the energy flow to and from a unit area of the boundary. The situation for an incident P-wave is shown in Fig. 4. By elementary wave theory it can be shown that the energy transmitted per unit of time through a unit area of the wave front of an S-wave with amplitude $B$ is

$$W_s = \frac{1}{2} \rho \, V_s \, \omega^2 \, B^2$$  \hspace{1cm} (24)$$

The similar expression for a P-wave with amplitude $A$ is

$$W_p = \frac{1}{2} \rho \, V_s \, \omega^2 \, A^2$$  \hspace{1cm} (25)$$

The area of the wave front for the incoming P-wave in Fig. 4 is $\sin \theta$ and the
shown in Fig. 7 and Eqs. 24 and 25 give the following counterparts to Eqs. 26 and 27

\[ E_t = \frac{1}{2} \rho V_S \omega^2 \sin \nu \]  \hspace{1cm} (29)

\[ E_r = \frac{1}{2} \rho V_S \omega^2 \sin \nu \cdot B^2 + \frac{1}{2} \rho V_S \omega^2 \sin \theta A^2 \]  \hspace{1cm} (30)

and the energy ratio becomes

\[ \frac{E_x}{E_l} = B^2 + \frac{1}{s} \sin \theta \sin \nu A^2 \]  \hspace{1cm} (31)

for \( \nu \geq \nu_{cr} \). This equation is valid only in the region \( \nu > \nu_{cr} \). For the subcritical region Eq. 30 must be replaced by

\[ E_r = \frac{1}{2} \rho V_S \omega^2 \sin \nu (B_1^2 + B_2^2) \]  \hspace{1cm} (32)

Eq. 32 does not include energy carried by the boundary wave because this wave transmits no energy to or from the boundary. The energy ratio becomes

\[ \frac{E_x}{E_l} = B_1^2 + B_2^2 \]  \hspace{1cm} (33)

The variation of the energy ratio with the angle of incidence for the case \( a = b = 1 \), and \( \mu = 0.25 \) is shown in Fig. 8. The curve is similar to Fig. 5 determined for the incident P-wave, and it may be shown that the viscous boundary defined by \( a = b = 1 \) is 95% effective in absorbing S-waves. It is important to note that the boundary has not been accounted for in complete detail, and some irregularities must be expected due to the existence of this wave. A more detailed study has shown that the relative effect of the boundary wave decreases with increasing frequency and increasing length of the boundary.

In summary it has been shown that the viscous boundary defined by

\[ \sigma = \rho V_P \dot{w} \]  \hspace{1cm} (34)

and

\[ \tau = \rho V_S \dot{u} \]  \hspace{1cm} (35)

is a nearly perfect absorber of harmonic elastic waves. Because the absorption characteristics are independent of frequency, the boundary can absorb both harmonic and nonharmonic waves. The viscous boundary defined by Eqs. 34 and 35 will be referred to as the standard viscous boundary.

Rayleigh Wave Absorption.—For steady state problems it is possible to design a viscous boundary that can completely absorb Rayleigh waves. The required boundary is similar to the standard viscous boundary defined in Eqs. 1 and 2 except that the parameters \( a \) and \( b \) vary with the distance from the free surface.

Consider a Rayleigh wave travelling with velocity \( V_R \) in the positive \( x \)-direction along the surface of the half space shown in Fig. 9. As shown by Ewing, Jardetzky and Press (2), the displacements are

\[ u = f(kz) \sin (\omega t - kz) \]  \hspace{1cm} (36)

\[ w = g(kz) \cos (\omega t - kz) \]  \hspace{1cm} (37)

in which the wave number \( k \) is defined by

\[ k = \frac{\omega}{V_R} \]  \hspace{1cm} (38)

For the special case of a homogeneous half space the functions \( f(kz) \) and \( g(kz) \) vary as shown in Fig. 10(a). The velocity of the Rayleigh wave may be expressed as a fraction of \( V_S \) by

\[ V_R = \frac{V_S}{\eta} \]  \hspace{1cm} (39)

The value of \( \eta \) and the functions \( f(kz) \) and \( g(kz) \) vary with Poisson's ratio. For \( \mu = 1/4 \) they are \( \eta = 1.08766 \) and

\[ f(kz) = D [\exp(-0.8475 \, kz) - 0.5773 \exp(-0.3933 \, kz)] \]  \hspace{1cm} (40)
in which \( D \) is a constant.

The compressive stress on a vertical plane is

\[
\sigma = -(\lambda + 2G) \frac{\partial u}{\partial x} - \lambda \frac{\partial v}{\partial z}
\]  

\[ \text{(42)} \]

in which \( \lambda = 2G \mu/(1 - 2\mu) \) is Lamé's constant. Substitution of Eqs. 36 and 37 yields

\[
\sigma = k \left[ (\lambda + 2G) f'(kz) - \lambda g'(kz) \right] \cos(\omega t - kz)
\]  

\[ \text{(43)} \]

in which the notation \( g'(kz) \) indicates the differentiation, \( dg/d(kz) \). Similarly the shear stress on a vertical plane is

\[
\tau = -k G \left[ f'(kz) + g'(kz) \right] \sin(\omega t - kz)
\]  

\[ \text{(44)} \]

The particle velocities are by simple differentiation of Eqs. 36 and 37

\[
\dot{u} = \omega f(kz) \cos(\omega t - kz)
\]  

\[ \text{(45)} \]

\[
\dot{w} = -\omega g(kz) \sin(\omega t - kz)
\]  

\[ \text{(46)} \]

Perfect energy absorption will be obtained if Eqs. 1 and 2 are satisfied identically. The required values of \( a \) and \( b \) may therefore be found by simple substitution of Eqs. 43 to 46 into Eqs. 1 and 2. The result is

\[
a(kz) = \frac{1}{\rho V_p} \frac{\partial}{\partial u}
\]  

\[ \text{(47)} \]

\[
b(kz) = \frac{1}{\rho V_s} \frac{\tau}{\omega} = \eta \left[ 1 + \frac{f'(kz)}{g(kz)} \right]
\]  

\[ \text{(48)} \]

The variation of \( a \) and \( b \) with \( kz \) is shown in Fig. 10(b). Recognizing that the physical meaning of the variable \( kz \) is \( 2\pi \times \text{depth/wavelength} \) it can be seen that at depths greater than \( \pi \) wave length the parameters \( a \) and \( b \) approach constant values. At the depth where the horizontal displacement vanishes the parameter \( a \) goes to infinity which agrees with the physical fact that an infinitely viscous dashpot is required to fix a point.

**NUMERICAL METHOD**

The proposed method has been applied to the foundation vibration problem defined in Fig. 11. The system is axisymmetrical about a vertical axis and consists of a weightless, perfectly rigid, frictionless, circular footing resting on a homogeneous, elastic half space. The footing is excited by a vertical harmonic force in the axis of symmetry. This problem has the advantage that an independent solution is available [see Lysmer and Richart (3)] which may be used to check the accuracy of the solution obtained by the method proposed herein.

The vertical displacement \( \delta \) of the footing is

\[
\delta = \frac{P_o}{K} F e^{i\omega t}
\]  

\[ \text{(49)} \]

where \( P_o \) and \( \omega \) = the amplitude and frequency, respectively, of the exciting

**FIG. 11.—FOUNDATION VIBRATION PROBLEM**

A finite model of the infinite system was formed by replacing the far field with the standard viscous boundary defined in Eqs. 34 and 35 and is shown in Fig. 13(a). The response of the resulting finite system cannot be found by analytical methods but may be approximated by the response of a finite element system of the type shown in Fig. 13(b).

Finite Element Method.—The finite element method is an established technique of numerical analysis and only details germane to the problem at hand will be presented herein. The equation of motion for the system in Fig. 13(b) is

\[
[M] \{\ddot u\} + [C] \{\dot u\} + [K] \{u\} = \{P\}
\]  

\[ \text{(52)} \]
respectively. The mass matrix [M] is diagonal because all masses are lumped at the nodal points. The stiffness matrix [K] is identical to the static stiffness matrix for the case where all nodal points are free and may be determined by standard methods. The linear strain rectangular element described by Costantini (1) was used. The damping matrix [C] is a diagonal matrix containing the lumped coefficients for the dashpots on the viscous boundary.

The solution to the steady state problem may be found by the method of complex response. The force vector is

\[ \{P\} = \{P_0\} e^{i\omega t} \quad (53) \]

in which \(\{P_0\}\) is a constant vector. The response is

\[ \{u\} = \{U\} e^{i\omega t} \quad (54a) \]

\[ \{u\} = i \omega \{U\} e^{i\omega t} \quad (54b) \]

and \(\{\dot{u}\} = -\omega^2 \{U\} e^{i\omega t} \quad (54c)\)

in which \(\{U\}\) is a constant complex displacement vector. By substitution of
The displacement vector \( \{u\} \) contains the footing displacement \( \hat{6} \); thus, Eqs. 49 and 50 can be used to compute the real and imaginary parts of the displacement function \( F = F_1 + i F_2 \) for comparison with the solution for the infinite system.

Choice of Finite Element Mesh.—The accuracy of the finite element method depends on the ratio obtained by dividing the length of the side of the largest element by the minimum wavelength of elastic waves propagating in the system.

**TABLE 1.—COMPARISON OF DISPLACEMENT FUNCTIONS FOR SMOOTH AND ROUGH FOOTINGS**

<table>
<thead>
<tr>
<th>( a_0 )</th>
<th>( F_1 )</th>
<th>( F_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth footing</td>
<td>Rough footing</td>
<td>Smooth footing</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8312</td>
<td>0.8089</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5593</td>
<td>0.5604</td>
</tr>
<tr>
<td>1.5</td>
<td>0.3115</td>
<td>0.3327</td>
</tr>
</tbody>
</table>

All dimensions of the model to be reduced. These two effects tend to cancel each other; therefore, the accuracy is primarily a function of the total number of nodal points provided the element size is maintained within the subject criterion.

**NUMERICAL RESULTS**

Numerical results were obtained with two basic models: (1) A small model with 238 nodal points and a distance to the viscous boundary of 3/4 x the wavelength of Rayleigh waves; and (2) a large model with 682 nodal points and a distance to the boundary of 1-1/2 x the wavelength of Rayleigh waves. All calculations were carried out on a CDC 6400 computer. The total processing time required to obtain a value for \( F \) at one frequency was 2 min for the small model and 8 min for the large model.

**FIG. 15.—RESULTS WITH RAYLEIGH WAVE BOUNDARY**

For accurate results this ratio should be smaller than approximately 1/12. The accuracy depends on the distance from the excited zone to the viscous boundary due to imperfect wave absorption. The error decreases with an increasing ratio between this distance and the minimum wavelength.

The model must be accurate within the operating frequency range for a given problem. From the preceding remarks it is clear that an increase in frequency: (1) Requires a decrease in element size; and (2) allows the overall

**FIG. 16.—RESULTS FOR ROUGH, EMBEDDED FOOTINGS**

Comparison with Existing Solution.—Fig. 14 shows a comparison between the displacement function for the infinite system and values obtained utilizing the standard viscous boundary. Good agreement was found when the large model was used while only fair agreement was found for the values obtained by the small model.

A detailed study of the displacement fields obtained with the standard viscous boundary condition revealed minor irregularities caused by a reflected surface wave. The existence of this wave was to be expected because Miller and Purse (4) have shown that 87% of the energy radiating from a vertically vibrating footing is transmitted in the form of a Rayleigh wave and this type of wave cannot be completely absorbed by the standard viscous boundary. In an attempt to improve the accuracy, the standard viscous boundary on the
vertical face of the model in Fig. 13 was replaced by the Rayleigh wave absorbing boundary defined in Eqs. 47 and 48. The results of calculations based on this model are shown in Fig. 15, and indicates a significant gain in accuracy over the results obtained with the standard viscous boundary. It is, therefore, that the Rayleigh wave boundary could be used for all steady state half space problems. The Rayleigh wave boundary cannot be used for transient problems since it is frequency dependent.

Rough Versus Frictionless Footings.—The displacement function shown in Fig. 12 was derived on the assumption of a frictionless surface between the footing and the half space. Because most footings are poured directly on the ground it is of interest to study the effect of the introduction of a rough interface. This is easily accomplished by the finite element method and Table 1 shows the small difference between the displacement functions for a rough

and a frictionless footing. The large model with the Rayleigh wave boundary was used in all computations.

Embedded Foundations.—A more complicated case occurs when a rough rigid footing is embedded in the half space. This problem is easily solved by the proposed method due to the great flexibility of the finite element method. Computed displacement functions for two ratios of foundation depth to footing radius are shown in Fig. 16. In both cases the displacement functions are defined in Eqs. 49 and 50. Therefore, the difference between the curves for the embedded footings and the footing on the surface represents the error resulting from applying the theory for footings on the surface to embedded footings. The large model and the Rayleigh wave boundary condition were used in all computations.

FIG. 17.—STEADY STATE SPECTRA FOR EMBEDDED FOOTINGS

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It is convenient for design usage to present the preceding results in the form of response curves reflecting the effect of the mass of the footing. As shown by Lysmer and Richart (3) the displacement of a footing with mass \( m \) is

\[
\delta = \frac{F}{K} \frac{1}{1 - B \alpha_0^2 \rho} e^{i\omega t} \quad \cdots \quad (57)
\]

in which \( B = \frac{m}{\rho \gamma^2} \) \cdots \quad (58)

and \( K \) is defined in Eq. 50.

The displacement amplitude is

\[
\delta_{\text{max}} = \frac{F}{K} M \quad \cdots \quad (59)
\]

in which \( M \) is the response factor, expressed as

\[
M = \left| \frac{F}{1 - B \alpha_0^2 \rho} \right| = \sqrt{\frac{F_1^2 + F_2^2}{(1 - B \alpha_0^2 \rho) F_1^2 + (B \alpha_0^2 \rho)} \quad \cdots \quad (60)
\]

Design curves based on values of \( F_1 \) and \( F_2 \) obtained from Fig. 16 are presented in Fig. 17.

CONCLUSIONS

A numerical method for the analysis of dynamic problems involving infinite continuous systems has been presented. Results obtained by the proposed method show excellent agreement with available theoretical results for an axisymmetric foundation vibration problem. Two previously unsolvable problems analyzed by the new method were: (1) The effect of friction between a vibrating footing and its foundation; and (2) the effect of embedment of a vibrating footing into the subsoil. The latter effect is quite pronounced and a set of response curves for design use are presented.

The analytical method described has applications far beyond the simple cases presented herein. It may be applied to any infinite system for which all disturbances and irregular geometrical features are limited to a small region of an otherwise homogeneous and linearly elastic space. The irregular zone need not be linear and the exciting forces need not be harmonic; they should have a vanishing time average, however, because the viscous boundary cannot support a static load. Problems involving nonlinearities and transient loads obviously cannot be handled by the method of complex vectors used in this paper but they may be solved by step-by-step procedures.

Only elastic solids have been considered in this paper, but with some modification, it should be possible to adapt the proposed method for the analysis of the near field around antennae and acoustic generators.

ACKNOWLEDGMENTS

The National Science Foundation provided the financial support for this investigation through the Research Initiation Grant No. NSF-GK-1467 to the
University of California at Berkeley. The writers are grateful for this support. All computations were performed on the CDC 6400 digital computer at the University of California at Berkeley.

APPENDIX I.—REFERENCES


APPENDIX II.—NOTATION

The following symbols are used in this paper:

\[ A \] = amplitude of reflected P-wave (see Fig. 2);
\[ A_1, A_2 \] = real and imaginary parts of \( A \);
\[ a \] = dimensionless viscous coefficient related to normal stress (see Eq. 1);
\[ a_o \] = frequency ratio (see Eq. 51);
\[ B \] = amplitude of reflected S-wave (see Fig. 2), or mass ratio (see Eq. 58);
\[ B_1, B_2 \] = real and imaginary parts of \( B \);
\[ b \] = dimensionless viscous coefficient related to shear stresses (see Eq. 2);
\[ [C] \] = damping matrix (see Eq. 52);
\[ c \] = velocity of wave front along boundary (see Eq. 7);
\[ D \] = arbitrary constant (see Eq. 40);
\[ E_i \] = energy of incident waves (see Eq. 26);
\[ E_r \] = energy of reflected waves (see Eq. 27);
\[ e \] = 2.718 \ldots = base of natural logarithm;
\[ F \] = displacement function (see Eq. 49);
\[ F_1, F_2 \] = real and imaginary parts of \( F \);
\[ f(kz) \] = amplitude function for the horizontal component of Rayleigh wave (see Eq. 36);
\[ G \] = shear modulus;
\[ g(kz) \] = amplitude function for the vertical component of Rayleigh wave (see Eq. 37);
\[ H \] = depth of embedment (see Fig. 17);
\[ i \] = imaginary unit;
\[ K \] = spring constant (see Eq. 50);
\[ [K] \] = stiffness matrix (see Eq. 52);
\[ [K] \] = modified complex stiffness matrix (see Eq. 58);
\[ k \] = wave number (see Eqs. 12 and 38);
\[ M \] = response factor (see Eq. 60);
\[ [M] \] = mass matrix (see Eq. 52);
\[ m \] = mass of footing (see Fig. 17);
\[ \{P\} \] = force vector, time dependent (see Eq. 52);
\[ P_o \] = amplitude of harmonic force on footing (see Fig. 11);
\[ \{P_o\} \] = force vector, time independent (see Eq. 53);
\[ \tau_o \] = radius of footing (see Fig. 11);
\[ s \] = ratio between the velocities of S-waves and P-waves (see Eq. 5);
\[ t \] = time;
\[ \{U\} \] = displacement vector, time independent (see Eq. 54);
\[ u, \dot{u}, \ddot{u} \] = displacement and velocity in \( x \)-direction;
\[ \{u\}, \{\dot{u}\}, \{\ddot{u}\} \] = displacement, velocity and acceleration vector (see Eq. 52);
\[ V_p \] = velocity of P-waves (see Eq. 4);
\[ V_r \] = velocity of Rayleigh waves (see Eq. 39);
\[ V_s \] = velocity of S-waves (see Eq. 3);
\[ W_p \] = intensity of P-waves (see Eq. 25);
\[ W_s \] = intensity of S-waves (see Eq. 24);
\[ w, \dot{w}, \ddot{w} \] = displacement and velocity in \( z \)-direction;
\[ x \] = horizontal coordinate;
\[ z \] = vertical coordinate;
\[ \delta \] = complex vertical displacement of footing (see Eq. 49);
\[ \delta_{max} \] = absolute value of \( \delta \) = amplitude (see Eq. 59);
\[ \eta \] = ratio between velocities of S-waves and Rayleigh waves (see Eq. 39);
\[ \theta \] = angle between P-wave ray and boundary (see Fig. 2);
\[ \lambda \] = Lame's constant (see Eq. 42);
\[ \mu \] = Poisson's ratio;
\[ \nu \] = angle between S-wave ray and boundary (see Fig. 3);
\[ \nu_{cr} \] = critical value of \( \nu \) (see Eq. 21);
\[ \pi \] = 3.1416 \ldots ;
\[ \rho \] = mass density;
\[ \sigma \] = normal stress;
\[ \tau \] = shear stress;
\[ \phi \] = wave potential for P-waves (see Eq. 8);
\[ \psi \] = wave potential for S-waves (see Eq. 9); and
\[ \omega \] = circular frequency.