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Probabilistic Elasto-Plasticity: Formulation in 1D

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Abstract A second-order exact expression for evolution of Probability Density Function (PDF) of stress is derived for general, one dimensional (1-D) elastic-plastic constitutive rate equations with uncertain material parameters. The Eulerian–Lagrangian (EL) form of Fokker–Planck–Kolmogorov (FPK) equation is used for this purpose. It is also shown that by using EL form of FPK, the so called "closure problem" associated with regular perturbation methods used so far, is resolved too. The use of EL form of FPK also replaces repetitive and computationally expensive deterministic elastic-plastic computations associated with Monte Carlo technique.

The derived general expression are specialized to the particular cases of point location scale linear elastic and elastic–plastic constitutive equations, related to associated Drucker-Prager with linear hardening

In a companion paper, the solution of FPK equations for 1D is presented, discussed and illustrated through number of examples.

1 Introduction

Advanced elasto–plasticity constitutive models, when properly calibrated, are very accurate in capturing important aspects of material behavior. However, all materials', and in particular geomaterials' (soil, rock, concrete, powder, bone etc.) behavior is uncertain due to inherent spatial and point-wise uncertainties. These uncertainties in material properties could outweigh the advantages gained by using advanced constitutive models. For example, Fig. 1 shows a schematic of anticipated influence of material uncertainties on a bi-linear elastic-plastic stress-strain behavior. Depending on uncertainties in material properties and interaction between them, the behavior of the same material could be very different.

The uncertainties in material properties are inevitable in real materials and it is best to account for them in modeling and simulation. In traditional deterministic constitutive modeling, material models are calibrated against set of experimental data. Although those experimental data sets generally exhibit statistical distribution, the models are usually calibrated against the mean of the data and all the information about uncertainties is neglected.

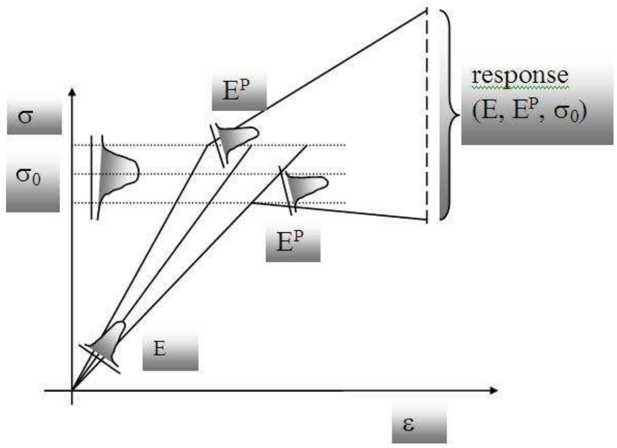

Fig. 1 Anticipated Influence of Material Fluctuations on Stress-Strain Behavior

The modeling and simulation of solids and structures with uncertain material properties involves two steps: (a) classification and quantification of uncertainties and (b) propagation of uncertainties through governing differential equations.

The uncertainties can be broadly classified into aleatory and epistemic types. Aleatory uncertainties are associated with the inherent variabilities of nature. This type of uncertainty can not be reduced. Highly developed mathematical theory is available for dealing with aleatory uncertainty. On the other hand, epistemic uncertainties arise due to our lack of knowledge. This type of uncertainty can be reduced by collecting more data but the mathematical tools to deal with them are not highly developed (e.g. fuzzy logic Zadeh (1983), convex models Ben-Haim and Elishakoff (1990), interval arithmetic Moore (1979) etc.). Hence, it proves useful to trade epistemic uncertainties for aleatory uncertainties in order to facilitate their propagation through the governing equations using advanced mathematical tools. It is important to note that in trading-off epistemic uncertainties for aleatory uncertainties, one doesn't reduce the total uncertainties in the system, but assumes that the uncertainties in the system are irreducible. Under the framework of probability theory, uncertain material parameters are modeled as random variables or random fields (Vanmarcke, 1983). We note recent works in quantifying the uncertainties in material (soil) properties for geotechnical engineering applications, Lumb (1966), Vanmarcke (1977), Mayerhoff (1993), DeGroot and Baecher (1993), Popescu (1995), Lacasse and Nadim (1996), Popescu et al. (1998), Phoon and Kulhawy (1999a,b), Fenton (1999a,b), Duncan (2000), Rackwitz (2000), Marosi and Hiltunen (2004), and Stokoe II et al. (2004) The issue of uncertain material properties becomes very pronounced when one starts dealing with the boundary value problems with uncertain material properties (elastic or elastic–plastic).

In mechanics, the equilibrium equation,$A\sigma = \phi(t)$, together with the strain compatibility equation, $Bu = \epsilon$, and the constitutive equation, $\sigma = D\epsilon$, are sufficient[1] to describe the behavior of the solid. Rigorous mathematical theory has been developed for problems where the only random parameter is the external force $\phi(t)$. In this case, the probability distribution function (PDF) of the response process will satisfy FPK partial differential

[1] Generalized stress is σ, $\phi(t)$ is generalized forces that can be time dependent, u is generalized displacements, ϵ is generalized strain and A, B, and D are operators which could be linear or non-linear.

equation (Soize, 1994) . With appropriate initial and boundary conditions the FPK PDE can be solved for PDF of response variable. The numerical solution method for FPK equation by finite element method (FEM) is described by number of researchers e.g. Langtangen (1991), Masud and Bergman (2005).

The other extreme case, which is of interest in this work, is when the stochasticity of the system is purely due to operator uncertainty. Exact solution of the problems with stochastic operator was attempted by Hopf (1952) using characteristic functional approach. Later, Lee (1974) applied the methodology to the problem of wave propagation in random media and derived a FPK equation satisfied by the characteristic functional of the random wave field. This characteristic functional approach is very complicated for linear problems and becomes even more intractable (and possibly unsolvable) for nonlinear problems and problems with irregular geometries and boundary conditions.

Monte Carlo simulation technique is an alternative to analytical solution of partial differential equation with stochastic coefficient. Nice descriptions of different aspects of formulation of Monte Carlo technique for stochastic mechanics problem is described by Schueller (1997a). Monte Carlo method is very popular tool with the advantage that accurate solution can be obtained for any problem whose deterministic solution (either analytical or numerical) is known. Monte Carlo technique has been used by a number of researchers in obtaining probabilistic solution of geotechnical boundary value problems, e.g. Paice et al. (1996); Griffiths et al. (2002); Fenton and Griffiths (2003, 2005). Popescu et al. (1997), Mellah et al. (2000), De Lima et al. (2001), Koutsourelakis et al. (2002), Nobahar (2003). The major disadvantage of Monte Carlo analysis is the repetitive use of the deterministic model until the solution variable become statistically significant. The computational cost associated with it could be very high especially for non–linear problems with multiple uncertain material properties.

Various difficulties in finding analytical solutions and the high computational cost associated with Monte Carlo technique instigated development of numerical method for the solution of stochastic differential equation with random coefficient. For stochastic boundary value problems Stochastic Finite Element Method (SFEM) is the most popular such method. There exist several formulations of SFEM, among which perturbation (Kleiber and Hien (1992); Der Kiureghian and Ke (1988); Mellah et al. (2000); Gutierrez and De Borst (1999)) and Spectral (Ghanem and Spanos (2003); Keese and Matthies (2002); Xiu and Karniadakis (2003); Debusschere et al. (2003); Anders and Hori (2000)) methods are very popular. A nice review on advantages and disadvantages of different formulations of SFEM was provided by Matthies et al. (1997). Mathematical issues regarding different formulations of SFEM was addressed by Deb et al. (2001) and Babuska and Chatzipantelidis (2002). It is important to note that most of the formulations described in the above mentioned references are for geometrically linear or nonlinear problems with linear elastic material.

A limited number of references is also available related to geometric non–linear problems, Liu and Der Kiureghian (1991); Keese and Matthies (2002) and Keese (2003)). Similarly, there exist only few published references related to material non–linear (elastic–plastic) problems with uncertain material parameters. The major difficulty in extending the available formulations of SFEM to general elastic–plastic problem is the high non–linear coupling in the elastic–plastic constitutive rate equation. First attempt to propagate uncertainties through elastic–plastic constitutive equations considering random Young's modulus was published only recently, e.g. Anders and Hori (1999, 2000). The perturbation expansion at the stochastic mean behavior (considering only the first term of the expansion) was used for the material nonlinear part in the above mentioned references. In computing the mean behavior the Authors took the advantage of bounding media approximation. Although this method doesn't suffer from computational difficulty associated with Monte Carlo method for problems having no closed-form solution, it inherits "closure problem" and the "small coefficient of variation" requirements for the material parameters. Closure problem refers to the need for higher order statistical moments in order to calculate lower order statistical moments Kavvas (2003). The small COV requirement claims that the perturbation method can be used (with reasonable accuracy) for probabilistic simulations of solids and structures with uncertain properties only if their COV $<$ 20 % (Sudret and Der Kiureghian, 2000). For soils and other natural materials, COVs are rarely below 20 % (Lacasse and Nadim

(1996); Phoon and Kulhawy (1999a,b)). Furthermore, with bounding media approximation, difficulty arises in computing the mean behavior when one considers uncertainties in internal variable(s) and/or direction(s) of evolution of internal variable(s).

The focus of present work is on development of methodology for the probabilistic simulation of constitutive behavior of elastic–plastic materials with uncertain properties. Recently, Kavvas (2003) obtained a generic Eulerian–Lagrangian (EL) form of FPK equation, exact to second-order, corresponding to any non–linear ordinary differential equation with random coefficients and random forcing. The approach using EL form of the FPK equation doesn't suffer from the drawbacks of Monte Carlo method and perturbation technique. In this paper the authors applied developed EL form of the FPK equation to obtain probabilistic formulation for a general, one-dimensional incremental elastic–plastic constitutive equation with random coefficient. The solution methodology is designed with several applications in mind, namely to

- obtain probabilistic stress–strain behavior from spatial average form (up-scaled form) of constitutive equation, when input uncertain material properties to the constitutive equation are random fields; and
- obtain probabilistic stress-strain behavior from point-location scale constitutive equation, when input uncertain material properties to the constitutive equation are random variables.

Application of the developed methodology is demonstrated on a particular point-location scale one-dimensional constitutive equation, namely Drucker–Prager associative linear hardening elastic–plastic material model. In this paper, derivation is made of the EL form of FPK equation that govern the 1D probabilistic elastic–plastic material models with uncertain material parameters. This general formulation is then specialized to a particular 1D Drucker–Prager associative linear hardening material model. It is noted that the solution of FPK equations can be quite computationally intensive. However, in presented case, with 1D constitutive problem, this solution poses no significant computational burden. It is also noted that the solutions of FPK in stress–strain space inherently represents a Markov process. In other words, there is no probability of switching to alternate equilibrium branch (say softening branch with localization of deformation) as Markov process requires that paths in probability density space are strictly followed from initial conditions all through the loading process.

In the companion paper the solution methodology of the FPK equation corresponding to Drucker–Prager associative linear hardening material model is described, along with illustrative examples. The methodology is general enough that it allows extension to three-dimensions and incorporation into a general stochastic finite element framework. This work is underway and will be reported in future publications.

2 General Formulation

The incremental form of spatial-average elastic-plastic constitutive equation can be written as

$$\frac{d\sigma_{ij}(x_t,t)}{dt} = D_{ijkl}(x_t,t)\frac{d\epsilon_{kl}(x_t,t)}{dt} \tag{1}$$

where x_t is current spatial location at time t, and the continuum stiffness tensor $D_{ijkl}(x_t,t)$ can be either elastic or elastic-plastic

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & ;\ f < 0 \vee (f = 0 \wedge df < 0) \\ \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn}\dfrac{\partial U}{\partial \sigma_{mn}}\dfrac{\partial f}{\partial \sigma_{pq}}D^{el}_{pqkl}}{\dfrac{\partial f}{\partial \sigma_{rs}}D^{el}_{rstu}\dfrac{\partial U}{\partial \sigma_{tu}} - \dfrac{\partial f}{\partial q_*}r_*} & ;\ f = 0 \vee df = 0 \end{cases} \tag{2}$$

and where D^{el}_{ijkl} is the elastic stiffness tensor, D^{ep}_{ijkl} is the elastic–plastic continuum stiffness tensor, f is the yield function, which is a function of stress (σ_{ij}) and internal variables (q_*), U is the plastic potential function (also a function of stress and internal variables). The internal variables (q_*) could be scalar(s) (for perfectly-plastic and isotropic hardening models), second-order tensor (for translational and rotational kinematic hardening) or fourth-order tensor (for distortional hardening). Therefore, the most general form of incremental constitutive equation in terms of its parameters can be written as

$$\frac{d\sigma_{ij}(x_t,t)}{dt} = \beta_{ijkl}(\sigma_{ij}, D_{ijkl}, q_*, r_*; x_t, t)\frac{d\epsilon_{kl}(x_t,t)}{dt} \tag{3}$$

Due to randomnesses in elastic constants (D^{el}_{ijkl}) and internal variables (q_*) and/or rate of evolution of internal variables (r_*) the material stiffness operator β_{ijkl} in Eq. (3) becomes stochastic. It follows that the Equation (1) becomes a linear/non-linear ordinary differential equations with stochastic coefficients. Similarly, randomness in he forcing term (ϵ_{kl}) of Equation (3) results in Equation (3) becoming linear/non-linear ordinary differential equations with stochastic forcing. This can be generalized, so that randomnesses in material properties and forcing function of Equation (3) results in Equation (3) becoming a linear/non-linear ordinary differential equation with stochastic coefficients and stochastic forcing.

In order to gain better understanding of the effects of random material parameters and forcing on response, focus is shifted from a general 3D case to a 1D case. In what follows, the probabilistic formulation for 1-D constitutive elastic–plastic incremental equation with stochastic coefficient and stochastic forcing is derived. In addition to that, the probabilistic formulation for 1-D elastic linear constitutive equation is obtained as a special case of non-linear general derivation.

Focusing on 1-D behavior, the Eq. (3) is written as

$$\frac{d\sigma(x_t, t)}{dt} = \beta(\sigma, D, q, r; x_t, t)\frac{d\epsilon(x_t, t)}{dt} \tag{4}$$

which is a non-linear ordinary differential equation with stochastic coefficient and stochastic forcing. The right hand side of Eq. (4) is replaced with the function η as

$$\eta(\sigma, D, q, r, \epsilon; x, t) = \beta(\sigma, D, q, r; x_t, t)\frac{d\epsilon(x_t, t)}{dt} \tag{5}$$

so that now Eq. (4) can be written as

$$\frac{\partial \sigma(x_t, t)}{\partial t} = \eta(\sigma, D, q, r, \epsilon; x, t) \tag{6}$$

with initial condition,

$$\sigma(x, 0) = \sigma_0 \tag{7}$$

In the above Eq. (6) σ can be considered to represent a point in the σ-space and hence, the Eq. (6) determines the velocity for the point in that σ-space. This may be visualized, from the initial point, and given initial condition σ_0, as a trajectory that describes the corresponding solution of the non-linear stochastic ordinary differential equation (ODE) (Eq. (6)). Considering now a cloud of initial points (refer to Fig. 2), described by a density $\rho(\sigma, 0)$ in the σ-space.

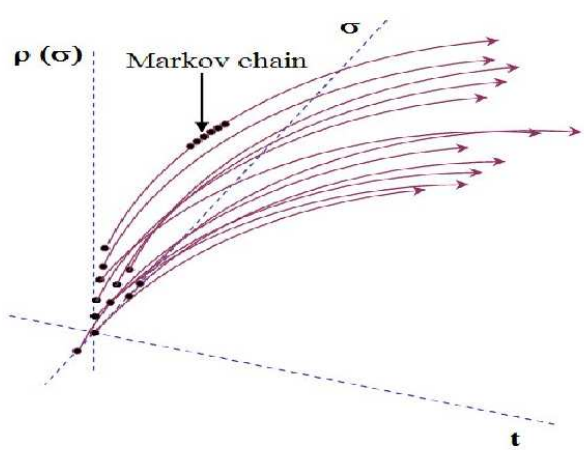

Fig. 2 Movements of Cloud of Initial Points, described by density $\rho(\sigma, 0)$, in the σ-space

The phase density ρ of $\sigma(x, t)$ (movement of any point dictated by Eq. (6)) varies in time according to a continuity equation which expresses the conservation of all these points in the σ-space. This continuity equation can be expressed in mathematical terms, using Kubo's stochastic Liouville equation (Kubo, 1963):

$$\frac{\partial \rho(\sigma(x,t),t)}{\partial t} = -\frac{\partial}{\partial \sigma} \eta[\sigma(x,t), D(x), q(x), r(x), \epsilon(x,t)].\rho[\sigma(x,t), t] \tag{8}$$

with an initial condition,

$$\rho(\sigma, 0) = \delta(\sigma - \sigma_0) \tag{9}$$

where δ is the Dirac delta function and Eq. (9) is the probabilistic restatement in the σ-phase space of the original deterministic initial condition (Eq. (7)). Here it proves useful to recall Van Kampen's Lemma (Van Kampen, 1976), which states that the ensemble average of a phase density is the probability density

$$< \rho(\sigma, t) >= P(\sigma, t) \tag{10}$$

where, the symbol $< \cdot >$ denotes the expectation operation, and $P(\sigma, t)$ denotes evolutionary probability density of the state variable σ of the constitutive rate equation (Eq. (4)).

In order to obtain the deterministic[2] probability density function (PDF) (σ, t) of the state variable, σ, it is necessary to obtain the deterministic partial differential equation (PDE) of the σ-space mean phase density $<$

[2] The probability density function describes probability of occurrence of certain event, but is a deterministic quantity.

$\rho(\sigma, t) >$ from the linear stochastic PDE system (Eqs. (8) and (9)). This necessitates the derivation of the ensemble average form of Eq. (8) for $< \rho(\sigma, t) >$.

This ensemble average was recently derived as (for detailed derivation, refer to Kavvas and Karakas (1996))

$$\frac{\partial \left\langle \rho(\sigma(x_t, t), t) \right\rangle}{\partial t} = - \frac{\partial}{\partial \sigma} \left[\left\{ \Xi(x,t) - \int_0^t d\tau \Pi(x,t,\tau) \right\} \left\langle \rho(\sigma(x_t, t), t) \right\rangle \right] + \frac{\partial}{\partial \sigma} \left[\int_0^t d\tau \Omega(x,t,\tau) \frac{\partial \left\langle \rho(\sigma(x_t, t), t) \right\rangle}{\partial \sigma} \right] \tag{11}$$

to exact second order (to the order of the covariance time of η). In Eq. (11), $\Xi(x, t)$ is the ensemble average of the function η at time t and is given as,

$$\Xi(x,t) = \left\langle \eta(\sigma(x_t, t), D(x_t), q(x_t), r(x_t), \epsilon(x_t, t)) \right\rangle ;$$

and $\Pi(x, t, \tau)$ is the covariance between the function η at time t and the derivative of the function η with respect to stress, σ, at time $t - \tau$ and is given as,

$$\Pi(x,t,\tau) = Cov_0 \left[\eta(\sigma(x_t, t), D(x_t), q(x_t), r(x_t), \epsilon(x_t, t)); \frac{\partial \eta(\sigma(x_{t-\tau}, t-\tau), D(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau))}{\partial \sigma} \right] ;$$

while $\Omega(x, t, \tau)$ is the covariance between function η at time t and function η at time $t - \tau$ and is given as,

$$\Omega(x,t,\tau) = Cov_0 \left[\eta(\sigma(x_t, t), D(x_t), q(x_t), r(x_t), \epsilon(x_t, t)); \eta(\sigma(x_{t-\tau}, t-\tau), D(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)) \right]$$

It is important to note that the covariances appearing in the above equations are time ordered covariances, which for two dummy variables $p(x, t_1)$ and $p(x, t_2)$ can be written as,

$$Cov_0\left[p(x,t_1); p(x,t_2)\right] = \langle p(x,t_1)p(x,t_2)\rangle - \langle p(x,t_1)\rangle \cdot \langle p(x,t_2)\rangle \tag{12}$$

By combining Eqs. (10) and (11) and rearranging the terms yields the following Eulerian–Lagrangian form of second-order accurate Fokker-Planck equation [3] (FPE) (for details, refer to Kavvas (Kavvas, 2003)):

$$\begin{aligned}
&\frac{\partial P(\sigma(x_t,t),t)}{\partial t} = \\
&- \frac{\partial}{\partial \sigma}\left[\left\{\Xi(x,t) + \int_0^t d\tau \Phi(x,t,\tau)\right\} P(\sigma(x_t,t),t)\right] \\
&+ \frac{\partial^2}{\partial \sigma^2}\left[\int_0^t d\tau \Omega(x,t,\tau) P(\sigma(x_t,t),t)\right]
\end{aligned} \tag{13}$$

where, $\Phi(x,t,\tau)$ is the time-ordered covariance between the derivative of the function η with respect to stress, σ at time t and the function η at time $t - \tau$ and is given as,

$$\begin{aligned}
\Phi(x,t,\tau) = Cov_0 \Bigg[&\frac{\partial \eta(\sigma(x_t,t), D(x_t), q(x_t), r(x_t)\epsilon(x_t,t))}{\partial \sigma}; \\
&\eta(\sigma(x_{t-\tau}, t-\tau), D(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)) \Bigg]
\end{aligned}$$

Eq. (13) is the most general relation for probabilistic behavior of inelastic (non–linear, elastic–plastic) 1-D stochastic incremental constitutive equation. The solution of this deterministic linear FPE (Eq. (13)), in terms of the probability density $P(\sigma, t)$, under appropriate initial and boundary

[3] also known as Forward–Kolmogorov Equation or Fokker–Planck–Kolmogorov (FPK) equation (Risken, 1989), (Gardiner, 2004), (Schueller, 1997a)

conditions will yield the PDF of the state variable σ of the original 1-D non-linear stochastic constitutive rate equation (Eq. (4)). It is important to note that while the original equation (Eq. (4)) is non-linear, the FPE (Eq. (13)) is linear in terms of its unknown, the probability density $P(\sigma, t)$ of the state variable σ. This linearity, in turn, provides significant advantages in the solution of the probabilistic behavior of the incremental constitutive equation (Eq. (4)).

One should also note that Eq. (13) is a mixed Eulerian-Lagrangian equation. This stems from the fact that while the real space location x_t at time t is known, the location $x_{t-\tau}$ is an unknown. If one assumes small strain theory, one can relate the unknown location $x_{t-\tau}$ from the known location x_t by using the strain rate, $\dot{\epsilon}$ (=$d\epsilon/dt$) as,

$$x_{t-\tau} = (1 - \dot{\epsilon}\tau)x_t \tag{14}$$

Once the probability density function $P(\sigma, t)$ is obtained it can be used to obtain the mean of state variable (σ) by usual expectation operation

$$< \sigma(t) >= \int \sigma(t) P(\sigma(t)) d\sigma(t) \tag{15}$$

Another possible way to obtain the mean of state variable is to use the equivalence between FPE and Itô–operator stochastic differential equation (Gardiner, 2004). In this case Itô–operator stochastic differential equation equivalent to Eq. (13) is

$$
\begin{aligned}
d\sigma(x,t) = & \Bigg\{ \Big\langle \eta(\sigma(x_t,t), D(x_t), q(x_t), r(x_t), \epsilon(x_t,t)) \Big\rangle \\
& + \int_0^t d\tau Cov_0 \left[\frac{\partial \eta(\sigma(x_t,t), D(x_t), q(x_t), r(x_t), \epsilon(x_t,t))}{\partial \sigma}; \right. \\
& \left. \eta(\sigma(x_{t-\tau}, t-\tau), D(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)) \right] \Bigg\} dt \\
& + b(\sigma,t) dW(t)
\end{aligned}
\tag{16}
$$

where,

$$
\begin{aligned}
b^2(\sigma,t) = 2 \int_0^t d\tau Cov_0 & \Big[\eta(\sigma(x_t,t), D(x_t), q(x_t), r(x_t), \epsilon(x_t,t)); \\
& \eta(\sigma(x_{t-\tau}, t-\tau), D(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)) \Big]
\end{aligned}
\tag{17}
$$

and, $dW(t)$ is an increment of Wiener process W with $< dW(t) >= 0$. It is also interesting to note that all the stochasticity of the original equation (Eq. (4)) are lumped together in the last term (Wiener increment term) of the right-hand-side of Eq. (16). By taking advantage of the independent increment property of the Wiener process ($< dW(t) >= 0$), one can derive the differential equation which describes the evolution of mean of state variable (σ) of the nonlinear constitutive rate equation in time and space as, (e.g. (Kavvas, 2003))

$$
\begin{aligned}
\frac{< d\sigma(x,t) >}{dt} = & \Big\langle \eta(\sigma(x_t,t), D(x_t), q(x_t), r(x_t), \epsilon(x_t,t)) \Big\rangle \\
& + \int_0^t d\tau Cov_0 \left[\frac{\partial \eta(\sigma(x_t,t), D(x_t), q(x_t), r(x_t), \epsilon(x_t,t))}{\partial \sigma}; \right. \\
& \left. \eta(\sigma(x_{t-\tau}, t-\tau), D(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)) \right]
\end{aligned}
\tag{18}
$$

Eq. (18) is a nonlocal integro-differential equation in Eulerian-Lagrangian form, since, although the location x_t at time t is known, the Lagrangian location $x_{t-\tau}$ is an unknown which is determined by Eq. (14). It is important to note that the state variable appearing within $\eta(\cdot)$ on the right-hand-side of Eq. (18) is random and needs to be treated accordingly.

This concludes the development of relation for probabilistic behavior of 1-D elastic–plastic constitutive incremental equation with stochastic coefficients and stochastic forcing in most general form. In the following section the developed general relation is specialized to two particular types of point-location scale constitutive modeling: a) 1-D (shear) linear elastic constitutive behavior, and b) 1-D (shear) elastic-plastic Drucker-Prager associative linear hardening constitutive behavior.

3 Linear Elastic Probabilistic 1-D Constitutive Incremental Equation

The 1-D (shear) point-location scale linear elastic constitutive rate equation can be written as

$$\frac{d\sigma_{12}}{dt} = G \ \frac{d\epsilon_{12}}{dt} \tag{19}$$

so that the function η, defined in Eq. (5), can be written as

$$\eta = G \ \frac{d\epsilon_{12}}{dt} \tag{20}$$

and hence, considering both the shear modulus, G and the strain rate, $d\epsilon_{12}/dt(t)$ as random, one can substitute Eq. (20) in Eq. (13) to obtain the particular FPK equation for the probabilistic behavior of 1-D (shear) point-location scale linear elastic constitutive incremental equation as,

$$\begin{aligned}
&\frac{\partial P(\sigma_{12}(t),t)}{\partial t} = \\
&- \frac{\partial}{\partial \sigma_{12}} \left[\left\{ \left\langle G \frac{d\epsilon_{12}}{dt}(t) \right\rangle \right. \right. \\
&+ \left. \left. \int_0^t d\tau Cov_0 \left[\frac{\partial}{\partial \sigma_{12}} \left(G \frac{d\epsilon_{12}}{dt}(t) \right) ; G \frac{d\epsilon_{12}}{dt}(t-\tau) \right] \right\} P(\sigma_{12}(t),t) \right] \\
&+ \frac{\partial^2}{\partial \sigma_{12}^2} \left[\left\{ \int_0^t d\tau Cov_0 \left[G \frac{d\epsilon_{12}}{dt}(t) ; G \frac{d\epsilon_{12}}{dt}(t-\tau) \right] \right\} P(\sigma_{12}(t),t) \right]
\end{aligned} \tag{21}$$

The first random process in the covariance term of the first coefficient on the r.h.s of above equation (Eq. (21)) is independent of σ_{12} and by noting that covariance of zero with any random process is zero, one can further simplify the above equation as,

$$\begin{aligned}
&\frac{\partial P(\sigma_{12}(t),t)}{\partial t} = \\
&- \frac{\partial}{\partial \sigma_{12}} \left[\left\langle G \frac{d\epsilon_{12}}{dt}(t) \right\rangle P(\sigma_{12}(t),t) \right] \\
&+ \frac{\partial^2}{\partial \sigma_{12}^2} \left[\left\{ \int_0^t d\tau Cov_0 \left[G \frac{d\epsilon_{12}}{dt}(t) ; G \frac{d\epsilon_{12}}{dt}(t-\tau) \right] \right\} P(\sigma_{12}(t),t) \right]
\end{aligned} \tag{22}$$

Given the random shear modulus and shear strain rate random process, with appropriate initial and boundary conditions, Eq. (22) will predict the evolution of probability density function of shear stress with time following 1-D linear elastic shear constitutive incremental equation. Once the PDF of shear stress is obtained, one can integrate it to obtain mean (ensemble average) and covariance of the temporal-spatial random field of shear stress. It should be noted that there exist several other exact methods (e.g. cumulant expansion method) to obtain probabilistic behavior of stochastic processes driven by linear ordinary differential equations. The main objective of FPK based approach presented here is to deal with non-linear stochastic processes (e.g. elastic-plastic constitutive rate equation) and the linear elastic case is obtained as a special case of that non-linear process.

4 Elastic–Plastic Probabilistic 1-D Constitutive Incremental Equation

For materials obeying Drucker-Prager yield criteria (without cohesion), the yield surface can be written as:

$$f = \sqrt{J_2} - \alpha I_1 \tag{23}$$

where $J_2 = \frac{1}{2} S_{ij} s_{ij}$ is the second invariant of the deviatoric stress tensor $s_{ij} = \sigma_{ij} - 1/3\delta_{ij}\sigma_{kk}$, and $I_1 = \sigma_{ii}$ is the first invariant of the stress tensor, and α, an internal variable, is a function of friction angle ($\alpha = 2\sin(\phi)/(\sqrt(3)(3 - sin\phi))$, where ϕ is the friction angle (e.g. (Chen and Han, 1988)).

By assuming associative flow rule, so that the yield function has the same derivatives as the plastic flow function

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial U}{\partial \sigma_{ij}} \tag{24}$$

one can expand parts of the tangent constitutive tensor given in Eq. (2) (from Page 10), to read[4]

$$
\begin{aligned}
A_{kl} = \frac{\partial f}{\partial \sigma_{pq}} D_{pqkl} = \quad & A_{kl} \frac{\partial f}{\partial I_1} \left(G \left(\frac{\partial I_1}{\partial \sigma_{11}} \delta_{1l} \delta_{1k} + \frac{\partial I_1}{\partial \sigma_{22}} \delta_{2l} \delta_{2k} + \frac{\partial I_1}{\partial \sigma_{33}} \delta_{3l} \delta_{3k} \right) \right. \\
& \left. + \left(K - \frac{2}{3} G \right) \frac{\partial I_1}{\partial \sigma_{cd}} \delta_{cd} \delta_{kl} \right) \\
& + \frac{\partial f}{\partial \sqrt{J_2}} \left(G \frac{\partial \sqrt{J_1}}{\partial \sigma_{ij}} \delta_{ik} \delta_{jl} + \left(K - \frac{2}{3} G \right) \frac{\partial \sqrt{J_2}}{\partial \sigma_{ab}} \delta_{ab} \delta_{kl} \right)
\end{aligned}
\tag{25}
$$

and,

$$
\begin{aligned}
B = & \frac{\partial f}{\partial \sigma_{rs}} D_{rstu} \frac{\partial f}{\partial \sigma_{tu}} = \\
& \left(\frac{\partial f}{\partial I_1} \right)^2 \left(G \left(\left(\frac{\partial I_1}{\partial \sigma_{11}} \right)^2 + \left(\frac{\partial I_1}{\partial \sigma_{22}} \right)^2 + \left(\frac{\partial I_1}{\partial \sigma_{33}} \right)^2 \right) \right. \\
& + \left(K - \frac{2}{3} G \right) \left(\frac{\partial I_1}{\partial \sigma_{ij}} \delta_{ij} \right)^2 \\
& \left. + \left(\frac{\partial f}{\partial \sqrt{J_2}} \right)^2 \left(G \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} + \left(K - \frac{2}{3} G \right) \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} \delta_{ij} \right)^2 \right) \right)
\end{aligned}
\tag{26}
$$

where, K and G are the elastic bulk modulus and the elastic shear modulus respectively.

By further assuming that the evolution of internal variable is a function of equivalent plastic strain[5], $e^p_{eq} = 2/3 e^p_{ij} e^p_{ij}$ then one can write

$$
K_P = -\frac{\partial f}{\partial q_n} \, r_n = -\frac{1}{\sqrt{3}} \frac{\partial f}{\partial \alpha} \frac{d\alpha}{de^p_{eq}} \frac{\partial f}{\partial \sqrt{J_2}}
\tag{27}
$$

[4] A more detailed derivation of this probabilistic differentiation is given in the Appendix.

[5] This is a fairly common assumption, e.g. (Chen and Han, 1988)

It should be noted that since material properties are assumed to be random, the resulting stress tensor will also become random and hence the derivatives of the stress invariants with respect to stress tensor (σ_{ij}) will become random. Therefore, differentiations appearing in Eqs. (25), (26), and (27) can not be carried out in a deterministic sense.

The parameter tensor in Eq. (1) then becomes

$$D^{ep}_{ijkl} = \begin{cases} G\delta_{ik}\delta_{jl} + \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{kl} & ;\ f < 0 \vee (f = 0 \wedge df < 0) \\ \\ G\delta_{ik}\delta_{jl} + \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{kl} - \frac{A_{ij}A_{kl}}{B + K_P} & ;\ f = 0 \vee df = 0 \end{cases} \tag{28}$$

where tensor A_{ij} and scalars B and K_P are defined by Eqs. (25), (26), and (27) respectively. The above equation (Eq. 28) represents a probabilistic continuum stiffness tensor for an elastic–plastic material model, in this case Drucker-Prager isotropic linear hardening material with associated plasticity. By focusing our attention on one dimensional point-location scale shear constitutive relationship between σ_{12} and ϵ_{12} for Drucker-Prager material model, one can simplify the function $\eta(\sigma, D, q, r, \epsilon; x, t)$ as defined in Eq. (5) (on Page 11) to read

$$\eta = \begin{cases} G\frac{d\epsilon_{12}}{dt} & ;\ f < 0 \vee (f = 0 \wedge df < 0) \\ \left(G - \frac{G^2\left(\frac{\partial f}{\partial\sqrt{J_2}}\frac{\sqrt{J_2}}{\partial\sigma_{12}}\right)^2}{B + K_P} \right)\frac{d\epsilon_{12}}{dt} & ;\ f = 0 \vee df = 0 \end{cases} \tag{29}$$

By considering both the material properties (shear modulus G, bulk modulus K, friction angle α, and rate of change of friction angle (linear hardening) α') and the strain rate ($d\epsilon_{12}/dt(t)$) as random, one can substitute η

as derived in Eq. (13) to obtain the particular FPK equation for the probabilistic behavior of Drucker-Prager associative linear hardening, 1-D point-location scale elastic-plastic shear constitutive rate equation. In particular, two cases are recognized, one for elastic (pre–yield) behavior of material $(f < 0 \vee (f = 0 \wedge df < 0))$

$$\begin{aligned}
&\frac{\partial P(\sigma_{12}(t),t)}{\partial t} = \\
&-\frac{\partial}{\partial \sigma_{12}}\left[\left\langle G\frac{d\epsilon_{12}}{dt}(t)\right\rangle P(\sigma_{12}(t),t)\right] \\
&+\frac{\partial^2}{\partial \sigma_{12}^2}\left[\left\{\int_0^t d\tau Cov_0\left[G\frac{d\epsilon_{12}}{dt}(t); G\frac{d\epsilon_{12}}{dt}(t-\tau)\right]\right\} P(\sigma_{12}(t),t)\right] \quad (30)
\end{aligned}$$

noting that this is the same equation as Eq. (22). In addition to that, the case of elastic–plastic behavior $(f = 0 \vee df = 0)$ is described by the following probabilistic equation

$$\begin{aligned}
&\frac{\partial P(\sigma_{12}(t),t)}{\partial t} = -\frac{\partial}{\partial \sigma_{12}}\left[\left\{\left\langle \left(G^{ep}(t)\right)\frac{d\epsilon_{12}}{dt}(t)\right\rangle\right.\right. \\
&+\left.\left.\int_0^t d\tau Cov_0\left[\frac{\partial}{\partial \sigma_{12}}\left(G^{ep}(t)\frac{d\epsilon_{12}}{dt}(t)\right); G^{ep}(t-\tau)\frac{d\epsilon_{12}}{dt}(t-\tau)\right]\right\} P(\sigma_{12}(t),t)\right] \\
&+\frac{\partial^2}{\partial \sigma_{12}^2}\left[\left\{\int_0^t d\tau Cov_0\left[G^{ep}(t)\frac{d\epsilon_{12}}{dt}(t); G^{ep}(t-\tau)\frac{d\epsilon_{12}}{dt}(t-\tau)\right]\right\} P(\sigma_{12}(t),t)\right]
\end{aligned} \quad (31)$$

where $G^{ep}(a)$ is defined as probabilistic elastic–plastic kernel and is introduced to shorten the writing (but will also have other uses later)

$$G^{ep}(a) = \left(G - \frac{G^2\left(\frac{\partial f}{\partial \sqrt{J_2}(a)}\frac{\partial \sqrt{J_2}(a)}{\partial \sigma_{12}(a)}\right)^2}{B(a) + K_P(a)} \right) \quad (32)$$

and a assumes values t or $t - \tau$.

It is important to note that the differentiations appearing in the coefficient terms of the FPK PDE (Eq. (31)), within the probabilistic elastic–plastic kernel $G^{ep}(a)$ (i.e. Eq. (32)), are for fixed values of σ_{12} and hence those differentiations can be carried out in a deterministic sense. After carrying out the differentiations, the probabilistic elastic–plastic kernel becomes

$$G^{ep}(a)|_{\sigma_{12}\rightarrow const.} = \left(G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1(a)\alpha'} \right) \tag{33}$$

which, after substitution, result in simplification of the FPK equation (31). Further simplification is possible by noting that the first random process in the covariance term of the first coefficient on the r.h.s of the equation (31) is independent of σ_{12}. Furthermore, since the covariance of zero with any random process is zero, the FPK equation (31) is further simplified to read

$$\begin{aligned} &\frac{\partial P(\sigma_{12}(t), t)}{\partial t} = \\ &- \frac{\partial}{\partial \sigma_{12}} \left[\left\langle G^{ep}(t) \frac{d\epsilon_{12}}{dt}(t) \right\rangle P(\sigma_{12}(t), t) \right] \\ &+ \frac{\partial^2}{\partial \sigma_{12}^2} \left[\left\{ \int_0^t d\tau Cov_0 \left[G^{ep}(t) \frac{d\epsilon_{12}}{dt}(t); G^{ep}(t-\tau) \frac{d\epsilon_{12}}{dt}(t-\tau) \right] \right\} P(\sigma_{12}(t), t) \right] \end{aligned} \tag{34}$$

where the probabilistic elastic–plastic kernel $G^{ep}(a)$ is given by the Eq. (33).

The evolution of a mean value of shear stress σ_{12} is obtained by substituting η (derived for Drucker-Prager material in Eq. (18)) as

$$\begin{aligned} \frac{< d\sigma_{12}(t) >}{dt} &= \left\langle G^{ep}(t) \frac{d\epsilon_{12}}{dt}(t) \right\rangle \\ &+ \int_0^t d\tau Cov_0 \left[\frac{\partial}{\partial \sigma_{12}} \left(G^{ep}(t) \frac{d\epsilon_{12}}{dt}(t) \right); G^{ep}(t-\tau) \frac{d\epsilon_{12}}{dt}(t-\tau) \right] \end{aligned} \tag{35}$$

It is important to note that the derivatives appearing in the mean and covariance term of the above Eulerian-Lagrangian integro-differential equation (Eq. (35) with the probabilistic elastic–plastic kernel defined through the Eq. (33)) are random differentiations and need to be treated accordingly. One possible approach to obtaining these differentiations could be perturbation with respect to mean (Anders and Hori, 2000) but the "closure problem" will appear. Hence, in this study the evolution of mean of σ_{12} will be obtained by the expectation operation on the PDF (Eq. (15)).

5 Initial and Boundary Conditions for the probabilistic elastic–plastic PDE

The PDE describing the probabilistic behavior of constitutive rate equations can be written in the following general form:

$$\begin{aligned}\frac{\partial P(\sigma_{12},t)}{\partial t} &= -\frac{\partial}{\partial \sigma_{12}}\left\{P(\sigma_{12},t)N_{(1)}\right\} + \frac{\partial^2}{\partial \sigma_{12}^2}\left\{P(\sigma_{12},t)N_{(2)}\right\} \\ &= -\frac{\partial}{\partial \sigma_{12}}\left[P(\sigma_{12},t)N_{(1)} - \frac{\partial}{\partial \sigma_{12}}\left\{P(\sigma_{12},t)N_{(2)}\right\}\right] \\ &= -\frac{\partial \zeta}{\partial \sigma_{12}} \end{aligned} \tag{36}$$

where, $N_{(1)}$ and $N_{(2)}$ are coefficients[6] of the PDE and represent the expressions within the curly braces of the first and second terms respectively on the right–hand–side of Eqs. (22), (30), and (31) These terms are called the advection ($N_{(1)}$) and diffusion ($N_{(2)}$) coefficients as the form of Eq. (36) closely resembles advection–diffusion equation (Gardiner, 2004). The symbol ζ in Eq. (36) can be considered to be the probability current. This

[6] Indices in brackets are not used in index summation convention.

follows from Eq. (36), which is a continuity equation and the state variable of the equation is probability density.

After introducing initial and boundary conditions, one can solve Eq. (36) for probability densities of σ_{12} with evolution of time. The initial condition could be deterministic or stochastic depending on the type of problem. For probabilistic behavior of linear elastic constitutive rate equation (Eq. (22)), one can assume that all the probability mass at time $t = 0$ is concentrated at $\sigma_{12} = 0$ or at some constant value of σ_{12} if there were some initial stresses to begin with (e.g. overburden pressure on a soil mass).

In mathematical term, this translates to,

$$P(\sigma_{12}, 0) = \delta(\sigma_{12}) \tag{37}$$

where, $\delta(\cdot)$ is the Dirac delta function.

For the post–yield behavior of probabilistic elastic-plastic constitutive rate Equation[7] (34), there will be a distribution of σ_{12}, corresponding to the solution of the pre-yield probabilistic behavior (Eq. (30)), to begin with. This probability mass ($P(\sigma_{12}(t), t)$), dictated by Eq. (13), will advect and diffuse into the domain (σ_{12}, t space) of the system throughout the evolution (in time/strain) of the simulation. Since it is required that the probability mass within the system is conserved i.e. no leaking is allowed at the boundaries, a reflecting barrier at the boundaries will be the preferred choice. In mathematical term, one can express this condition as (Gardiner, 2004)

$$\zeta(\sigma_{12}, t)|_{AtBoundaries} = 0 \tag{38}$$

[7] Specialized to Drucker-Prager associated linear hardening model.

In theory, the stress domain could extend from $-\infty$ to ∞ so that boundary conditions are then

$$\zeta(-\infty, t) = \zeta(\infty, t) = 0 \tag{39}$$

With these initial and boundary conditions, the probabilistic differential equation (with random material properties and random strain) for elasto–plasticity, specialized in this case to associated Drucker–Prager material model with linear hardening, and by using FPK transform described above, can be solved for probability densities of shear stress (σ_{12}) as it evolves with time/shear strain (ϵ_{12}).

6 Discussions and Conclusions

In this paper, a second-order exact expression for evolution of PDF of stress was derived for any general, one-dimensional spatial-average form elastic-plastic constitutive rate equation. The advantage of Fokker–Planck–Kolmogorov based approach as presented here is that it doesn't suffer from "closure problem" associated with regular perturbation approach nor it requires repetitive use of computationally expensive deterministic elastic-plastic model as associated with Monte Carlo simulation technique. Furthermore, the developed expression (Eq. (13)) is linear and deterministic PDE whereas the original constitutive rate equation (Eq. (4)) was stochastic and non-linear ODE. This deterministic linearity will in turn provide great simplicity in solving the PDE (Eq. (13)) for probability densities and subsequently mean and variance behavior of the state variable (stress) of the original ODE (Eq. (4)). An associated Drucker–Prager material model with uncertain parameters was used to illustrate the general form of probabilistic elastic and elastic–plastic equations on one dimensions.

In a companion paper, solution, discussion and illustrative examples are presented.

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7 Appendix

7.1 Derivation of A_{kl} (Eq. (25))

Chain rule of differentiation for yield function $f = f(I_1, J_2)$ is used to develop equation

$$\frac{\partial f}{\partial \sigma_{pq}} D_{pqkl} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{pq}} D_{pqkl} + \frac{\partial f}{\partial \sqrt{J_2}} \frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}} D_{pqkl} \tag{40}$$

The first term in the R.H.S of Eq. (40) can be written as:

$$\begin{aligned}\frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{pq}} D_{pqkl} &= \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{pq}} \left[G \delta_{pk} \delta ql + \left(K - \frac{2}{3} G \right) \delta_{pq} \delta_{kl} \right] \\ &= G \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{pq}} \delta_{pk} \delta_{ql} + \left(K - \frac{2}{3} G \right) \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{pq}} \delta_{pq} \delta_{kl}\end{aligned} \tag{41}$$

where now one can write the first term on the R.H.S of Eq. (41) as,

$$\begin{aligned}
\frac{\partial I_1}{\partial \sigma_{pq}}\delta_{pk}\delta_{ql} &= \left(\frac{\partial I_1}{\partial \sigma_{1q}}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{2q}}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{3q}}\delta_{3k}\right)\delta_{ql} \\
&= \left(\frac{\partial I_1}{\partial \sigma_{1q}}\delta_{ql}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{2q}}\delta_{ql}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{3q}}\delta_{ql}\right)\delta_{3k} \\
&= \left(\frac{\partial I_1}{\partial \sigma_{11}}\delta_{1l} + \frac{\partial I_1}{\partial \sigma_{12}}\delta_{2l} + \frac{\partial I_1}{\partial \sigma_{13}}\delta_{3l}\right)\delta_{1k} \\
&\quad + \left(\frac{\partial I_1}{\partial \sigma_{21}}\delta_{1l} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{2l} + \frac{\partial I_1}{\partial \sigma_{23}}\delta_{3l}\right)\delta_{2k} \\
&\quad + \left(\frac{\partial I_1}{\partial \sigma_{31}}\delta_{1l} + \frac{\partial I_1}{\partial \sigma_{32}}\delta_{2l} + \frac{\partial I_1}{\partial \sigma_{33}}\delta_{3l}\right)\delta_{3k} \\
&= \frac{\partial I_1}{\partial \sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{33}}\delta_{3l}\delta_{3k}
\end{aligned} \tag{42}$$

It is important to note that the differentiation is with respect to random stress so that terms in last line of previous equation cannot be simplified further. Similarly, from Eq. (41) it follows that

$$G\frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{pq}}\delta_{pk}\delta_{ql} = G\frac{\partial f}{\partial I_1}\left[\frac{\partial I_1}{\partial \sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{33}}\delta_{3l}\delta_{3k}\right] \tag{43}$$

where detailed derivations yield

$$\begin{aligned}
\frac{\partial I_1}{\partial \sigma_{pq}}\delta_{pq}\delta_{kl} &= \frac{\partial I_1}{\partial \sigma_{11}}\delta_{11}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{22}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{33}}\delta_{33}\delta_{kl} \\
&\quad +2\left(\frac{\partial I_1}{\partial \sigma_{12}}\delta_{12}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{23}}\delta_{23}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{31}}\delta_{31}\delta_{kl}\right) \\
&= \frac{\partial I_1}{\partial \sigma_{11}}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{33}}\delta_{kl}
\end{aligned} \tag{44}$$

so that the second part of the R.H.S. of Eq. (41)) can be written as

$$\left(K - \frac{2}{3}G\right)\frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{pq}}\delta pq\delta_{kl} = \left(K - \frac{2}{3}G\right)\frac{\partial f}{\partial I_1}\left[\frac{\partial I_1}{\partial \sigma_{11}}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{kl} + \frac{\partial I_1}{\partial \sigma_{33}}\right]\delta_{kl} \tag{45}$$

First part of the R.H.S. of the Eq. (40)) can be written as

$$\frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{pq}}D_{pqkl} = \frac{\partial f}{\partial I_1}G\left(\frac{\partial I_1}{\partial \sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{33}}\delta_{3l}\delta_{3k}\right) + \frac{\partial f}{\partial I_1}\left(K - \frac{2}{3}G\right)\left(\frac{\partial I_1}{\partial \sigma_{11}} + \frac{\partial I_1}{\partial \sigma_{22}} + \frac{\partial I_1}{\partial \sigma_{33}}\right)\delta_{kl} \tag{46}$$

while the second part of the same Eq. (40)) can be written as

$$\frac{\partial f}{\partial \sqrt{J_2}}\frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}}D_{pqkl} = \frac{\partial f}{\partial \sqrt{J_2}}\frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}}\left[G\delta_{pk}\delta ql + \left(K - \frac{2}{3}G\right)\delta pq\delta_{kl}\right] = G\frac{\partial f}{\partial \sqrt{J_2}}\frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}}\delta_{pk}\delta_{ql} + \left(K - \frac{2}{3}G\right)\frac{\partial f}{\partial \sqrt{J_2}}\frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}}\delta_{pq}\delta_{kl} \tag{47}$$

The second term of the previous equation (Eq. (47)) then becomes

$$\begin{aligned}\frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}}\delta_{pk}\delta_{ql} &= \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{1q}}\delta_{1k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{2q}}\delta_{2k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{3q}}\delta_{3k}\right)\delta_{ql} \\ &= \frac{\partial \sqrt{J_2}}{\partial \sigma_{1q}}\delta_{ql}\delta_{1k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{2q}}\delta_{ql}\delta_{2k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{3q}}\delta_{ql}\delta_{3k} \\ &= \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{11}}\delta_{1l} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{12}}\delta_{2l} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{13}}\delta_{3l}\right)\delta_{1k} \\ &+ \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{21}}\delta_{1l} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{22}}\delta_{2l} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{23}}\delta_{3l}\right)\delta_{2k} \\ &+ \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{31}}\delta_{1l} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{32}}\delta_{2l} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{33}}\delta_{3l}\right)\delta_{3k}\end{aligned} \tag{48}$$

which can be used to write the first term on the R.H.S. of Eq. (41)) as

$$\begin{aligned}G\frac{\partial f}{\partial \sqrt{J_2}}\frac{\partial \sqrt{J_2}}{\partial \sigma_{pq}}\delta_{pk}\delta_{ql} &= G\frac{\partial f}{\partial \sqrt{J_2}}\left[\frac{\partial \sqrt{J_2}}{\partial \sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{12}}\delta_{2l}\delta_{1k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{13}}\delta_{3l}\delta_{1k}\right] \\ &+ G\frac{\partial f}{\partial \sqrt{J_2}}\left[\frac{\partial \sqrt{J_2}}{\partial \sigma_{21}}\delta_{1l}\delta_{2k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{23}}\delta_{3l}\delta_{2k}\right] \\ &+ G\frac{\partial f}{\partial \sqrt{J_2}}\left[\frac{\partial \sqrt{J_2}}{\partial \sigma_{31}}\delta_{1l}\delta_{3k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{32}}\delta_{2l}\delta_{3k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{33}}\delta_{3l}\delta_{3k}\right]\end{aligned} \tag{49}$$

and since

$$\begin{aligned}\frac{\partial\sqrt{J_2}}{\partial\sigma_{pq}}\delta_{pq}\delta_{kl} &= \frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{11}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{22}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\delta_{33}\delta_{kl} \\ &+ 2\left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{12}}\delta_{12}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{23}}\delta_{23}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{31}}\delta_{31}\delta_{kl}\right) \\ &= \frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\delta_{kl} \end{aligned} \tag{50}$$

with

$$\left(K - \frac{2}{3}G\right)\frac{\partial f}{\partial\sqrt{J_2}}\frac{\partial\sqrt{J_2}}{\partial\sigma_{pq}}\delta pq\delta_{kl} = \left(K - \frac{2}{3}G\right)\frac{\partial f}{\partial\sqrt{J_2}}\left[\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\right]\delta_{kl} \tag{51}$$

and with

$$\begin{aligned}\frac{\partial f}{\partial\sqrt{J_2}}\frac{\partial\sqrt{J_2}}{\partial\sigma_{pq}}D_{pqkl} &= G\frac{\partial f}{\partial\sqrt{J_2}}\left[\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{12}}\delta_{2l}\delta_{1k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{13}}\delta_{3l}\delta_{1k}\right] \\ &+ G\frac{\partial f}{\partial\sqrt{J_2}}\left[\frac{\partial\sqrt{J_2}}{\partial\sigma_{21}}\delta_{1l}\delta_{2k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{23}}\delta_{3l}\delta_{2k}\right] \\ &+ G\frac{\partial f}{\partial\sqrt{J_2}}\left[\frac{\partial\sqrt{J_2}}{\partial\sigma_{31}}\delta_{1l}\delta_{3k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{32}}\delta_{2l}\delta_{3k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\delta_{3l}\delta_{3k}\right] \\ &+ \left(K - \frac{2}{3}G\right)\frac{\partial f}{\partial\sqrt{J_2}}\left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\right)\delta_{kl}\end{aligned} \tag{52}$$

will finally yield Eq. (40)),

$$\begin{aligned}\frac{\partial f}{\partial\sigma_{pq}}D_{pqkl} &= \frac{\partial f}{\partial I_1}\left[G\left(\frac{\partial I_1}{\partial\sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial I_1}{\partial\sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial I_1}{\partial\sigma_{33}}\delta_{3l}\delta_{3k}\right) + \left(K - \frac{2}{3}G\right)\frac{\partial I_1}{\partial\sigma_{cd}}\delta_{cd}\delta_{kl}\right] \\ &+ \frac{\partial f}{\partial\sqrt{J_2}}\left[G\frac{\partial\sqrt{J_1}}{\partial\sigma_{ij}}\delta_{ik}\delta_{jl} + \left(K - \frac{2}{3}G\right)\frac{\partial\sqrt{J_2}}{\partial\sigma_{ab}}\delta_{ab}\delta_{kl}\right]\end{aligned} \tag{53}$$

7.2 Derivation of B (Eq. (26))

Using Eq. (53), one can write:

$$
\begin{aligned}
\frac{\partial f}{\partial \sigma_{rs}} D_{rstu} \frac{\partial f}{\partial \sigma_{tu}} = \Bigg[& \frac{\partial f}{\partial I_1} G \left(\frac{\partial I_1}{\partial \sigma_{11}} \delta_{1l}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{22}} \delta_{2l}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{33}} \delta_{3l}\delta_{3k} \right) \\
& + \frac{\partial f}{\partial I_1} \left(K - \frac{2}{3} G \right) \left(\frac{\partial I_1}{\partial \sigma_{11}} \delta_{kl} + \frac{\partial I_1}{\partial \sigma_{22}} \delta_{kl} + \frac{\partial I_1}{\partial \sigma_{33}} \delta_{kl} \right) \\
& + G \frac{\partial f}{\partial \sqrt{J_2}} \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{11}} \delta_{1l}\delta_{1k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{12}} \delta_{2l}\delta_{1k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{13}} \delta_{3l}\delta_{1k} \right. \\
& + \frac{\partial \sqrt{J_2}}{\partial \sigma_{21}} \delta_{1l}\delta_{2k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{22}} \delta_{2l}\delta_{2k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{23}} \delta_{3l}\delta_{2k} \\
& \left. + \frac{\partial \sqrt{J_2}}{\partial \sigma_{31}} \delta_{1l}\delta_{3k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{32}} \delta_{2l}\delta_{3k} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{33}} \delta_{3l}\delta_{3k} \right) \\
& + \frac{\partial f}{\partial \sqrt{J_2}} \left(K - \frac{2}{3} G \right) \left(\frac{\partial \sqrt{J_2}}{\partial \sigma_{11}} \delta_{kl} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{22}} \delta_{kl} + \frac{\partial \sqrt{J_2}}{\partial \sigma_{33}} \delta_{kl} \right) \Bigg] \frac{\partial f}{\partial \sigma_{kl}}
\end{aligned}
\tag{54}
$$

where, by using

$$
\begin{aligned}
&\delta_{1l}\delta_{1k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{11}} \quad ; \quad \delta_{2l}\delta_{2k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{22}} \quad ; \quad \delta_{3l}\delta_{3k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{33}} \\
&\delta_{2l}\delta_{1k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{12}} \quad ; \quad \delta_{3l}\delta_{1k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{13}} \quad ; \quad \delta_{1l}\delta_{2k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{21}} \\
&\delta_{3l}\delta_{2k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{23}} \quad ; \quad \delta_{1l}\delta_{3k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{31}} \quad ; \quad \delta_{2l}\delta_{3k} \frac{\partial f}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{32}}
\end{aligned}
\tag{55}
$$

one can write the first part of the R.H.S. of Eq. (54) as

$$
\begin{aligned}
& \frac{\partial f}{\partial I_1} G \left(\frac{\partial I_1}{\partial \sigma_{11}} \delta_{1l}\delta_{1k} + \frac{\partial I_1}{\partial \sigma_{22}} \delta_{2l}\delta_{2k} + \frac{\partial I_1}{\partial \sigma_{33}} \delta_{3l}\delta_{3k} \right) \frac{\partial f}{\partial \sigma_{kl}} = \\
& = G \left[\left(\frac{\partial f}{\partial \sigma_{11}} \right)^2 + \left(\frac{\partial f}{\partial \sigma_{22}} \right)^2 + \left(\frac{\partial f}{\partial \sigma_{33}} \right)^2 \right] \\
& = G \left(\frac{\partial f}{\partial I_1} \right)^2 \left[\left(\frac{\partial I_1}{\partial \sigma_{11}} \right)^2 + \left(\frac{\partial I_1}{\partial \sigma_{22}} \right)^2 + \left(\frac{\partial I_1}{\partial \sigma_{33}} \right)^2 \right]
\end{aligned}
\tag{56}
$$

while the second part of the same equation (Eq. (54)) becomes

$$
\begin{aligned}
G\frac{\partial f}{\partial\sqrt{J_2}} & \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{1l}\delta_{1k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{12}}\delta_{2l}\delta_{1k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{13}}\delta_{3l}\delta_{1k}\right.\\
&+\frac{\partial\sqrt{J_2}}{\partial\sigma_{21}}\delta_{1l}\delta_{2k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{2l}\delta_{2k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{23}}\delta_{3l}\delta_{2k}\\
&\left.+\frac{\partial\sqrt{J_2}}{\partial\sigma_{31}}\delta_{1l}\delta_{3k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{32}}\delta_{2l}\delta_{3k} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\delta_{3l}\delta_{3k}\right)\frac{\partial f}{\partial\sigma_{kl}}\\
= G\left(\frac{\partial f}{\partial\sqrt{J_2}}\right)^2 & \left[\left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\right)^2 + \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{12}}\right)^2 + \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{13}}\right)^2\right.\\
&+\left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{21}}\right)^2 + \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\right)^2 + \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{23}}\right)^2\\
&\left.+\left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{31}}\right)^2 + \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{32}}\right)^2 + \left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\right)^2\right]
\end{aligned}
\tag{57}
$$

and since

$$
\frac{\partial f}{\partial\sigma_{kl}}\delta_{kl} = \frac{\partial f}{\partial\sigma_{11}} + \frac{\partial f}{\partial\sigma_{22}} + \frac{\partial f}{\partial\sigma_{33}}
\tag{58}
$$

it follows that

$$
\begin{aligned}
&\frac{\partial f}{\partial I_1}\left(K-\frac{2}{3}G\right)\left(\frac{\partial I_1}{\partial\sigma_{11}}\delta_{kl} + \frac{\partial I_1}{\partial\sigma_{22}}\delta_{kl} + \frac{\partial I_1}{\partial\sigma_{33}}\delta_{kl}\right)\frac{\partial f}{\partial\sigma_{kl}}\\
&= \left(K-\frac{2}{3}G\right)\left(\frac{\partial f}{\partial I_1}\right)^2\left[\frac{\partial I_1}{\partial\sigma_{11}} + \frac{\partial I_1}{\partial\sigma_{22}} + \frac{\partial I_1}{\partial\sigma_{33}}\right]^2
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
&\frac{\partial f}{\partial\sqrt{J_2}}\left(K-\frac{2}{3}G\right)\left(\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}}\delta_{kl} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\delta_{kl}\right)\frac{\partial f}{\partial\sigma_{kl}}\\
&= \left(K-\frac{2}{3}G\right)\left(\frac{\partial f}{\partial\sqrt{J_2}}\right)^2\left[\frac{\partial\sqrt{J_2}}{\partial\sigma_{11}} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{22}} + \frac{\partial\sqrt{J_2}}{\partial\sigma_{33}}\right]^2
\end{aligned}
\tag{60}
$$

Therefore, adding Eqs. (55), (56), (58), and (59) one can write:

$$B = \frac{\partial f}{\partial \sigma_{rs}} D_{rstu} \frac{\partial f}{\partial \sigma_{tu}} = \left(\frac{\partial f}{\partial I_1}\right)^2 \left[G \left\{ \left(\frac{\partial I_1}{\partial \sigma_{11}}\right)^2 + \left(\frac{\partial I_1}{\partial \sigma_{22}}\right)^2 + \left(\frac{\partial I_1}{\partial \sigma_{33}}\right)^2 \right\} + \left(K - \frac{2}{3}G\right) \left\{ \frac{\partial I_1}{\partial \sigma_{ij}} \delta_{ij} \right\}^2 \right] + \left(\frac{\partial f}{\partial \sqrt{J_2}}\right)^2 \left[G \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} + \left(K - \frac{2}{3}G\right) \left\{ \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} \delta_{ij} \right\}^2 \right] \quad (61)$$

7.3 Derivation of K_P (Eq. (27))

¿From the consistency condition ($\dot{f} = 0$), one can derive

$$K_P = -\frac{\partial f}{\partial q_n} r_n \quad (62)$$

with

$$\dot{q}_n = < L > r_n \quad (63)$$

where, $< \cdot >$ is the McCaulay bracket and L is the loading index given by

$$L = \frac{1}{K_P} L_{ij} \dot{\sigma}_{ij} \quad (64)$$

where, L_{ij} is a vector normal to the loading surface in stress space and dot represents time derivative. For Drucker-Prager yield criteria (Eq. (23)) one can simplify the above equation (Eq. (62)) as

$$K_P = -\frac{\partial f}{\partial \alpha} \bar{\alpha} \quad (65)$$

where,

$$\dot{\alpha} = < L > \bar{\alpha} \quad (66)$$

By assuming that the friction angle like internal variable α is a function of equivalent plastic strain (e^p_{eq}) i.e.

$$\alpha = \alpha(e^p_{eq}) \tag{67}$$

one can write, (using the relationship: $\dot{e}^p_{eq} = 1/\sqrt{3} < L > \partial f/\partial\sqrt{J_2}$)

$$\dot{\alpha} = \frac{d\alpha}{de^p_{eq}}\dot{e}^p_{eq} = \frac{1}{\sqrt{3}}\frac{d\alpha}{de^p_{eq}} < L > \frac{\partial f}{\partial\sqrt{J_2}} =< L > \bar{\alpha}$$

and therefore,

$$K_P = -\frac{\partial f}{\partial q_n}r_n = -\frac{\partial f}{\partial\alpha}\bar{\alpha} = -\frac{1}{\sqrt{3}}\frac{\partial f}{\partial\alpha}\frac{d\alpha}{de^p_{eq}}\frac{\partial f}{\partial\sqrt{J_2}} \tag{68}$$