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On Probabilistic Yielding of Materials

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SUMMARY

Uncertainty in material properties can have large effect on numerical modeling of solids and structures. This is particularly true as all natural and man made material exhibit spatial non-uniformity and point-wise uncertainty in material behavior.

Presented is the methodology that accounts for probabilistic yielding of elastic–plastic materials. The recently developed Eulerian–Lagrangian form of Fokker–Planck–Kolmogorov equation is used to obtain second order exact solution to elastic–plastic constitutive differential equations. In this paper that solution is used in deriving the weighted probabilities of elastic, elastic–plastic behavior and yielding. A number of examples for two commonly used material models, von Mises and Drucker–Prager, illustrated findings.

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1. INTRODUCTION

Elastic–plastic computations have so far been almost exclusively done in a deterministic fashion. This was and is still done, despite inherent uncertainty in material behavior. In particular, the elastic–plastic (inelastic) behavior of solids and structure is modeled using material parameters calibrated from a number of tests. Proper testing procedure would require a (statistically consistent) large number of tests, from which would then appropriate material parameters be calibrated. However, the ostensible claims of good economy usually dictate fewer tests, resulting in limited number of test results. Current state of the art is that those (few) test results are used to calibrate material parameters in a deterministic fashion, usually using mean of those few tests.

However, mechanical behavior of all engineering materials is inherently uncertain. The uncertain response follows from either spatial non-uniformity of material distribution or from inherent uncertainty of material behavior at the constitutive level. The uncertainty of material behavior propagates through numerical simulations of solids and structure. Deterministic simulations that are currently almost exclusively performed rely on safety factors, trying to take into account (material and modeling) uncertainties that were neglected. Recently, a number of methods were developed to deal with simulating solids and structures made of materials with random fields of material properties (Ghanem and Spanos [3], Matthies and Keese [8], Roberts and Spanos [10], Zhu et al, [15], Soize [13]). It should be noted that all of the previously mentioned references deal exclusively with elastic materials. Behavior of elastic–plastic materials with uncertain properties has not received much attention.

One of the earliest approaches to propagating randomness through the elastic–plastic constitutive equations (random Young's modulus) was presented by Anders and Hori [1]. They based their approach on perturbation expansion at the stochastic mean behavior and took advantage of bounding media analysis in computing the mean response. However, because of the use of Taylor series expansion in developments, this approach is limited to problems with small coefficients of variation (Sudret and Der Kiureghian [14]). Another disadvantage of the perturbation method is that it inherits the so-called "closure problem" (cf. Kavvas [7]), where information on higher–order moments is always needed to calculate lower–order moments. Similarly, Kaminski [6] used Monte–Carlo simulations with perturbation approach of stochastic finite element method for stochastic porous plasticity in solids. More recently, Fenton and Griffiths [2] used a Monte–Carlo method to propagate uncertainties through elastic–plastic $c - \phi$ soil. Monte–Carlo approach to accounting for uncertainties might be computationally expensive for elastic–plastic simulations. The method requires a statistically appropriate number of realizations per random variable in order to satisfy statistical accuracy. The Monte–Carlo simulation approach, however, finds a great use in verification of analytical developments (cf. Oberkampf et al, [9]).

A solution to overcoming drawbacks of Monte–Carlo technique and perturbation method is the use of general Eulerian-Lagrangian form of Fokker–Planck–Kolmogorov equation (FPKE) for the second–order exact probabilistic solution of non-linear ODE with stochastic coefficient and stochastic forcing (Kavvas [7]). Using the above mentioned Eulerian–Lagrangian form of FPKE, Jeremić et al. [5] and Sett et al. [11, 12] recently developed formulation and solution for the general 1-D elastic-plastic constitutive equation with random material properties and random strain rate.

One of the main advantages of FPKE based approach is that it provides a second order accurate probability density function (PDF) of stress (exact mean and variance) for given random material properties and/or random strain. In addition to that, for cases (material models) where FPKE is not solvable in closed-form solution, the deterministic linearity of the FPKE (with respect to the probability density of stress) considerably simplifies the numerical solution process.

In the above mentioned FPKE-based probabilistic elasto–plasticity papers (Jeremić et al. [5] and Sett et al. [11, 12]) the FPKE was solved separately (twice) for (probabilistic) elastic and then for (probabilistic) elastic–plastic phase of loading. The mean of yielding stress was used to make the separation, to decide on when the yielding occurs. However, since the material behavior is probabilistic, so should be the point (or rather region) of yielding. In this paper we derive the weighted probabilities of elastic or elastic–plastic behavior. These weighted probabilities are used to develop probabilistic elastic–plastic response of materials, that is based on probabilistic rather then mean yielding. Probabilistic yielding examples using von Mises and Drucker–Prager material models are used to illustrate developed methodology.

2. FPKE-based Probabilistic Elasto–Plasticity

The incremental form of spatial-average 3–D elastic-plastic constitutive rate equation can be written as:

$$\frac{d\sigma_{ij}(x_t,t)}{dt} = D^{ep}_{ijkl}(\sigma_{ij}, D^{el}_{ijkl}, f, U, q_*, r_*; x_t, t)\frac{d\epsilon_{kl}(x_t,t)}{dt} \tag{1}$$

where, D^{ep}_{ijkl} is the *random, non-linear elastic-plastic coefficient tensor* which is a function of random stress tensor (σ_{ij}), random elastic moduli tensor (D^{el}_{ijkl}), random yield function (f), random plastic potential function (U), random internal variables (q_*) and random direction of evolution of internal variables (r_*). The random internal variables (q_*) could be scalar (for perfectly plastic and isotropic hardening models), or second-order tensor (for translational and rotational kinematic hardening models), or fourth-order tensor (for distortional hardening models) or any combinations of the above. The same classification applies to the random direction of evolution of internal variables (r_*). By denoting all the random material parameters by a parameter tensor $D_{ijkl} = \left[D^{el}_{ijkl}, f, U, q_*, r_*\right]$ and by introducing a random operator tensor, η_{ij}, one can write Eq. (1) and initial conditions as,

$$\frac{d\sigma_{ij}(x,t)}{dt} = \eta_{ij}(\sigma_{ij}, D_{ijkl}, \epsilon_{kl}; x, t) \quad ; \quad \sigma_{ij}(x,0) = \sigma_{ij_0} \tag{2}$$

Using developments described in more details in Jeremić et al. [5] and Sett et al. [11], it can be shown that the probability density function (PDF) of stress $P(\sigma_{ij}(x_t,t),t)$, obeying Eq. (2), is governed by the Eulerian-Lagrangian form of Fokker–Planck–Kolmogorov equation of the form:

$$\begin{aligned}
\frac{\partial P(\sigma_{ij}(x_t,t),t)}{\partial t} &= \frac{\partial}{\partial \sigma_{mn}} \left[\left\{ \left\langle \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)) \right\rangle \right.\right. \\
&+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t))}{\partial \sigma_{ab}} ; \right. \\
&\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right] \\
&+ \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[\left\{ \int_0^t d\tau Cov_0 \left[\eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)); \right.\right.\right. \\
&\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right]
\end{aligned} \tag{3}$$

The above equation is second order accurate (mean value and variance) in probability density of stress state $P(\sigma_{ij}, t)$. The solution of this deterministic linear FPKE (Eq. (3)) in terms of $P(\sigma_{ij}, t)$ under appropriate initial and boundary conditions will yield the PDF of the state variable tensor σ_{ij}. It is important to note that while the original equation (Eq. (1)) is non-linear, the FPKE (Eq. (3)) is linear in terms of its unknown, the probability density $P(\sigma_{ij}, t)$ of the state variable tensor σ_{ij}. This linearity, in turn, provides significant advantages in probabilistic solution of the constitutive rate equation.

2.1. Specialization to 1D, von Mises and Drucker–Prager Hardening Material Models

The general, three dimensional PDF for stress (Eq. (3)) is specialized to point-location scale (where the uncertain material parameters are random variables) one dimensional case of shearing rate equations ($d\sigma_{xy}/dt = Gd\epsilon_{xy}/dt$) and is written for both linear elastic (using elastic G) and for elastic–plastic case (using elastic–plastic G):

$$\frac{\partial P(\sigma_{xy}(t))}{\partial t} = -\left\langle G\frac{d\epsilon_{xy}}{dt}\right\rangle \frac{\partial P(\sigma_{xy}(t))}{\partial \sigma_{xy}} + \left\{\int_0^t d\tau Cov_0\left[G\frac{d\epsilon_{xy}}{dt}; G\frac{d\epsilon_{xy}}{dt}\right]\right\} \frac{\partial^2 P(\sigma_{xy}(t))}{\partial \sigma_{xy}^2} \tag{4}$$

Previous equations can be written in generalized form describing evolution of stress (with appropriate initial and boundary conditions) as

$$\frac{\partial P}{\partial t} = -N_{(1)}\frac{\partial P}{\partial \sigma_{xy}} + N_{(2)}\frac{\partial^2 P}{\partial \sigma_{xy}^2} \quad ; \quad \zeta(-\infty, t) = \zeta(\infty, t) = 0 \tag{5}$$

where $N_{(1)}$ and $N_{(2)}$ are called advection and diffusion coefficients, respectively, and ζ is the probability current. The advection and diffusion coefficients $N_{(1)}$ and $N_{(2)}$ can be derived for any elastic and/or elastic–plastic material model. For example, for linear elastic material model, these coefficients are

$$N_{(1)}^{el} = \frac{d\epsilon_{xy}}{dt}\langle G\rangle \quad ; \quad N_{(2)}^{el} = t\left(\frac{d\epsilon_{xy}}{dt}\right)^2 Var[G] \tag{6}$$

Similarly, using equation (4), advection and diffusion coefficients for elastic–plastic von–Mises model (with hardening) are

$$N_{(1)}^{ep} = \frac{d\epsilon_{xy}}{dt}\left\langle G - \frac{G^2}{G + \frac{1}{\sqrt{3}}c'_u}\right\rangle \tag{7}$$

$$N_{(2)}^{ep} = t\left(\frac{d\epsilon_{xy}}{dt}\right)^2 Var\left[G - \frac{G^2}{G + \frac{1}{\sqrt{3}}c'_u}\right] \tag{8}$$

while these coefficients for elastic–plastic Drucker–Prager model (with hardening) are of the form

$$N^{ep}_{(1)} = \frac{d\epsilon_{xy}}{dt} \left\langle G - \frac{G^2}{G + 9K\alpha^2 + \frac{I_1\alpha'}{\sqrt{3}}} \right\rangle \tag{9}$$

$$N^{ep}_{(2)} = t \left(\frac{d\epsilon_{xy}}{dt}\right)^2 Var \left[G - \frac{G^2}{G + 9K\alpha^2 + \frac{I_1\alpha'}{\sqrt{3}}} \right] \tag{10}$$

Once the coefficients are derived, the solution to Eq. (5) can proceed using a number of different methods, including simple finite difference scheme.

2.2. Weighted Elastic and Elastic–Plastic Solution: Probabilistic Yielding

The above development needs solutions of two equations – one for the elastic (pre–yield) region, governed by $N^{el}_{(1)}$ and $N^{el}_{(2)}$ and the other for the elastic–plastic (post–yield) region, governed by $N^{ep}_{(1)}$ and $N^{ep}_{(2)}$ – for complete simulation of probabilistic constitutive behavior. However, if yield surface (yield point in 1–D) is uncertain, uncertainty propagates into separation of elastic and elastic–plastic regions. That is, depending on the degree of uncertainty of yield surface, stress points can only have certain probability of being in elastic or elastic–plastic state.

A solution to this problem of uncertain yielding is to assign weights to the elastic and elastic–plastic advection ($N^{el}_{(1)}$ and $N^{ep}_{(1)}$) and diffusion ($N^{el}_{(2)}$ and $N^{ep}_{(2)}$) coefficients based on the cumulative probability density function (CDF) of the yield function (or stress Σ_y in 1D) random variable. That combined (weighted) equation, which now has both elastic and elastic–plastic coefficient, appropriately weighted, can then be used to obtain the complete constitutive behavior with equivalent advection and diffusion coefficients. In other words, while solving the FPK partial differential equation, for each stress (σ) in the stress domain, probability weight will be assigned to the elastic and elastic–plastic advection and diffusion coefficients corresponding to that stress. Mathematically, the equivalent (weighted) advection and diffusion coefficients ($N^{eq}_{(1)}$ and $N^{eq}_{(2)}$) can be written as:

$$N^{eq}_{(1)}(\sigma) = (1 - P[\Sigma_y \leq \sigma])N^{el}_{(1)} + P[\Sigma_y \leq \sigma]N^{ep}_{(1)} \tag{11}$$

$$N^{eq}_{(2)}(\sigma) = (1 - P[\Sigma_y \leq \sigma])N^{el}_{(2)} + P[\Sigma_y \leq \sigma]N^{ep}_{(2)} \tag{12}$$

where $(1 - P[\Sigma_y \leq \sigma])$ represents the probability of material being elastic, while $P[\Sigma_y \leq \sigma]$ represents the probability of material being elastic–plastic.

For example, for a given CDF of yield stress (shown in Fig. 1(a)), the probability of yielding happening at $\sigma = 0.0012$ MPa[†] is $P[\Sigma_y \leq (\sigma = 0.0012 \text{ MPa})] = 0.8$ so that the equivalent advection and diffusion coefficients are

$$N^{eq}_{(1)}|_{\sigma=0.0012\ MPa} = (1 - 0.8)N^{el}_{(1)} + 0.8N^{ep}_{(1)}$$

$$N^{eq}_{(2)}|_{\sigma=0.0012\ MPa} = (1 - 0.8)N^{el}_{(2)} + 0.8N^{ep}_{(2)}$$

where linear elastic coefficients $N^{el}_{(1)}$ and $N^{el}_{(2)}$ are given by Eq. (6), while coefficients $N^{ep}_{(1)}$ and $N^{ep}_{(2)}$ are given by Eqs. (7) and (8) for von Mises and by Eqs. (9) and (10) for Drucker-Prager material models.

3. Examples

In this section the developed concept is applied to two elastic–plastic linear hardening material models. The models are represented by von Mises and Drucker–Prager yield and plastic potential functions, respectively, with linear isotropic hardening. For both models shear modulus and yield parameter (shear strength (c_u) for von Mises model and friction coefficient

[†]Mathematically we will write this probability as $P[\Sigma_y \leq (\sigma = 0.0012 \text{ MPa})]$

(α) for Drucker–Prager model) are considered independent, normally distributed random variables. Three examples are shown for each model. The first example for each material model presents a case where shear modulus and yield parameter are very uncertain. The other examples for each material model are limiting cases – one where shear modulus is fairly certain while yielding parameter is very uncertain, and the other where shear modulus is very certain while yield parameter is fairly certain. It is important to note that for all the examples, only one equation is solved, using equivalent advection and diffusion coefficients, to obtain both uncertain elastic and uncertain elastic–plastic behavior.

The governing FPK partial differential equation (Eq. (5)), with equivalent advection and diffusion coefficients (Eqs. (11) and (12)), was solved numerically using the *method of lines*. To this end, the FPK partial differential equation was first semi-discretized in the stress domain on a uniform grid by *central differences* to obtain a series of ordinary differential equation (ODE). These ODEs are then solved simultaneously, after incorporating boundary conditions, using a standard ODE solver which utilizes *ADAMS method* and *functional iteration*. Open source code SUNDIALS [4] was used for this purpose.

3.1. von Mises Associative, Linear Hardening Elastic–Plastic Model

The yielding probability weights assigned to the advection (N_1) and diffusion (N_2) coefficients are based on the cumulative density function (CDF) of the yield stress, which in this case corresponds to CDF of the shear strength (c_u)). The CDF of yield stress is shown in Fig. 1.

Fig. 2(a) shows the evolution of probability density function (PDF) of shear stress with respect to shear strain for probabilistic von Mises associative plasticity model. The shear modulus (G) is modeled as normally distributed random variable with a mean of 2.5 MPa

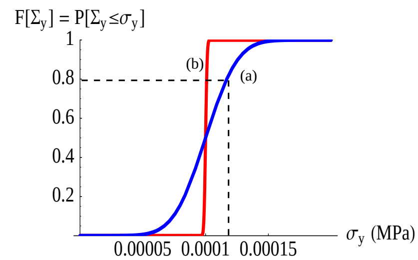


Figure 1. CDF of shear strength for von Mises model: (a) very uncertain case, (b) fairly certain case.

and coefficient of variation (COV) of 20 %. The shear strength (c_u) is also assumed to be normally distributed with a mean of 0.0001 MPa and COV of 20 %. The hardening parameter (c'_u, representing the rate of evolution of c_u with plastic strain) is considered deterministic with an assumed value of 0.3 MPa. The contours of the evolution of PDF of shear stress with shear strain are shown in Fig. 2(b), along with mean, mode (most probable solution), and the standard deviations. The deterministic solution obtained using mean values of shear

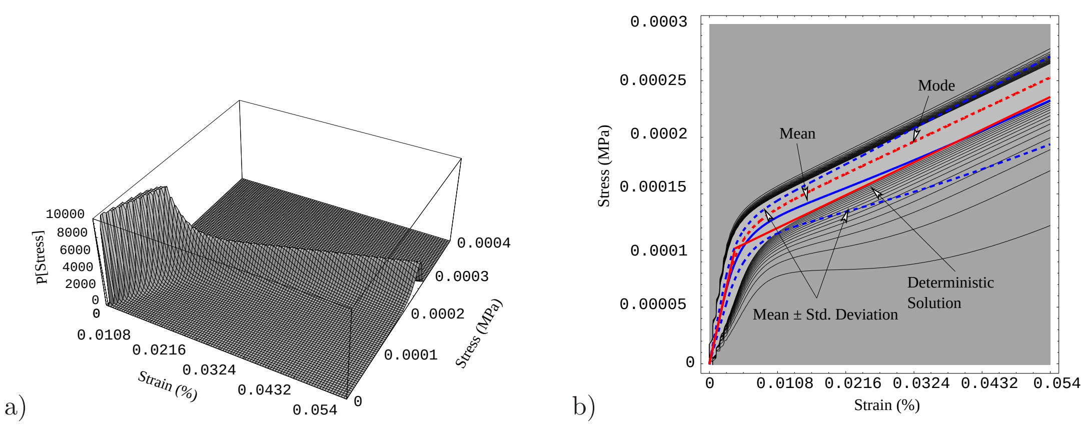


Figure 2. von Mises associative plasticity model with uncertain shear modulus and shear strength (yield parameter): (a) Evolution of PDF of stress with strain (PDF=10000 was used as a cutoff for surface plot) and (b) Contours of evolution of stress PDF with strain.

modulus (2.5 MPa) and shear strength (0.0001 MPa) is also shown. It is interesting to note the smooth response for probabilistic von Mises elasto–plastic model. On the other hand, the deterministic von Mises elastic–plastic model shows an expected sharp change in stiffness at the boundary between elastic and elastic–plastic solutions. In addition to that, the most likely stress response (mode) is different than the mean and the deterministic stresses. This difference in mode, mean and deterministic stress responses observed for linear hardening material model is a novel feature that deserves some attention. This, more realistic yielding response contrasts earlier observation of equivalence of mode, mean and deterministic solutions for probabilistic elasto–plasticity with mean stress yielding (Sett et al. [11]). In the approach presented here, involving probabilistic yielding, the difference between mode, mean and deterministic solutions is present even for linear elastic hardening cases (which will involve perfectly plastic material model as well).

The probabilistic solution for one of the limiting cases, where shear modulus is very uncertain (statistical properties are same as the above example), while shear strength is fairly certain (mean of 0.0001 MPa and COV of 1 %) is shown in Fig. 3(a). The CDF of the fairly certain yield stress is shown in Fig. 1. Note that, since yielding is fairly certain, the mean, mode and deterministic behaviors are very similar in the yielding region.

The other limiting case that was considered has a fairly certain shear modulus (mean of 2.5 MPa and COV of 1 %), however shear strength is very uncertain (mean of 0.0001 MPa and COV of 20 %). The simulation result is shown in Fig. 3(b). Note that, as shear modulus is fairly certain, the standard deviations lines almost match the mean and mode response line. The response is still probabilistic, if very sharp (in stress PDF – strain space) so instead of showing contours (most of which overlap) shown are only the mean, mode, standard deviation

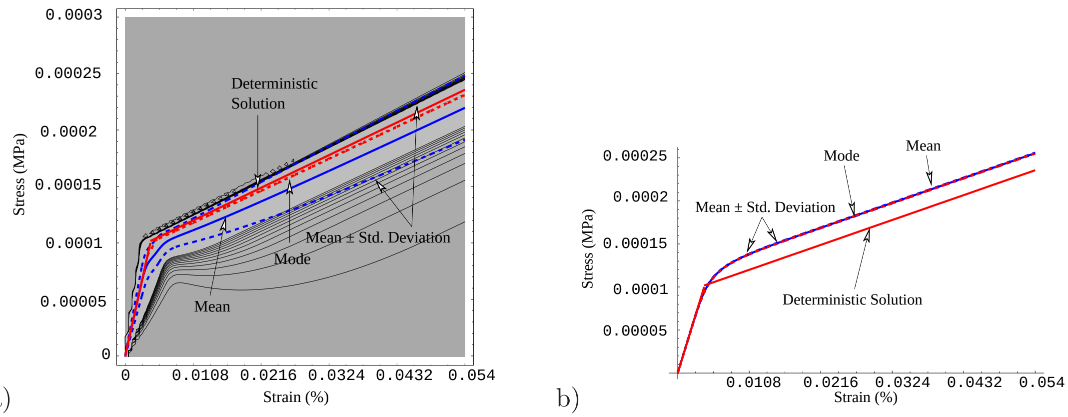


Figure 3. von Mises elastic–plastic model: (a) Shear modulus: very uncertain; shear strength: fairly certain, (b) Shear modulus: fairly certain; shear strength: very uncertain.

and the deterministic response. The certainty of response as shown in Fig. 3(b) is attributed to the lack of shear strength parameter c_u (which is the only very uncertain parameter here) in either of the coefficients $N^{ep}_{(1)}$ and $N^{ep}_{(2)}$ in equations (7) and (8). It is important to note, however, that the deterministic response is still quite a bit different from mode and mean response.

3.2. Drucker-Prager Associative Plasticity Model

Similar to the von Mises case, presented are results of probabilistic Drucker-Prager material model simulation, for the case where both shear modulus and yield parameter (frictional coefficient, α) are considered as very uncertain. The assumed mean and COV of normally distributed shear modulus (G) are 2.5 MPa and 20 %, respectively. The frictional coefficient (α) has a mean and COV of 0.1 and 20 %, respectively. The other parameters needed for Drucker–Prager probabilistic elastic–plastic simulations, were assumed deterministic and were as follows: the bulk modulus K = 3.33 MPa, the rate of evolution of α with plastic strain

$\alpha' = 5.5$, and the confinement pressure $I_1 = 0.01$ MPa. For a given confinement pressure (I_1) and (probabilistic) frictional coefficient (α), the CDF of a yield stress ($\sigma_y = I_1\alpha$) is calculated and shown in Fig. 4(a).

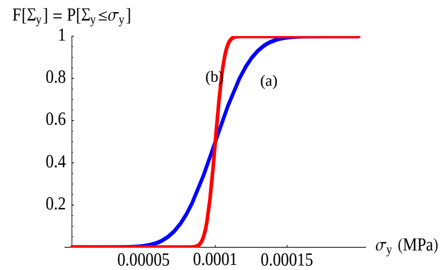
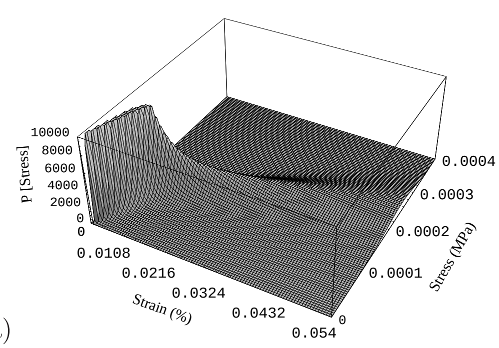


Figure 4. CDF of yield stresses for Drucker-Prager model: (a) very uncertain and (b) fairly certain

Fig. 5(a) shows the evolution of PDF of shear stress with strain. Same results are presented as contours of evolution of PDF of shear stress with shear strain along with the mean, mode, and the deterministic solution in Fig. 5(b).



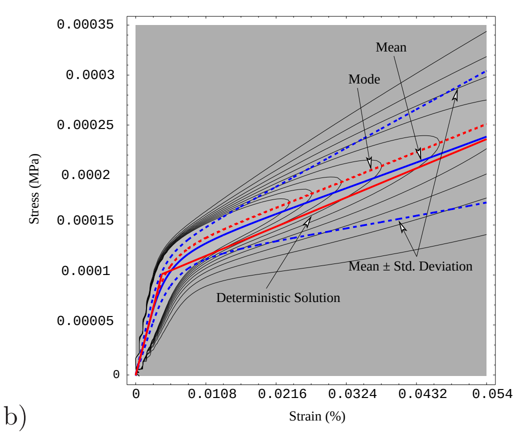


Figure 5. Drucker-Prager associative elastic–plastic model with uncertain shear modulus and frictional coefficient: (a) Evolution of probability density function (PDF) of stress with strain (PDF=10000 was used as a cutoff for surface plot) and (b) Contours of evolution of PDF with strain

The limiting cases, where shear modulus (G) is very uncertain (COV = 20 %), while frictional coefficient (α) is considered fairly certain (COV = 5 %) is shown in Figs. 6(a). This case is to be contrasted with the case where shear modulus (G) is fairly certain (COV = 1 %), while frictional coefficient (α) is very uncertain (COV = 20 %), shown in Figs. 6(b).

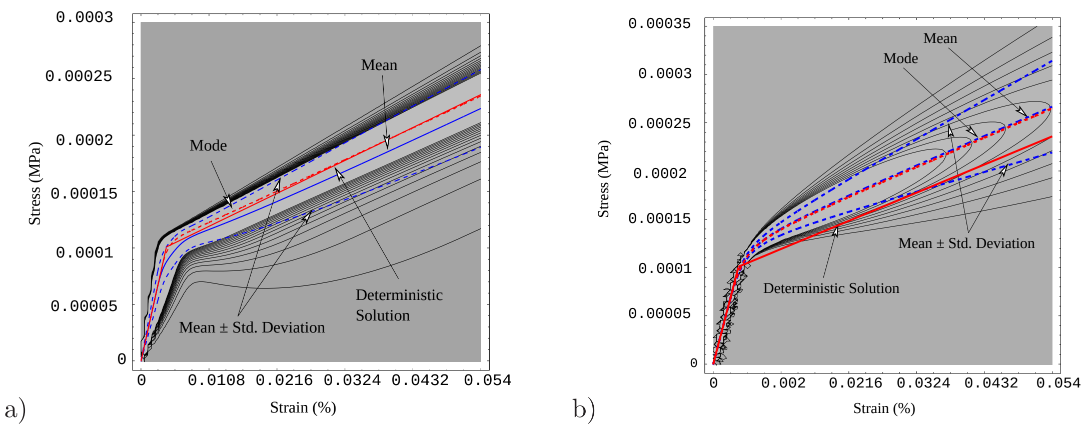


Figure 6. Drucker-Prager elastic–plastic model: (a) Shear modulus: very uncertain; frictional coefficient: fairly certain, (b) Shear modulus: fairly certain; frictional coefficient: very uncertain.

4. Summary

In this paper we presented the methodology that accounts for probabilistic yielding of elastic–plastic materials. Use was made of the recently developed second order accurate solution to the probabilistic elastic–plastic problem (which is based on solution of the Eulerian–Lagrangian form of Fokker–Planck–Kolmogorov equation). The derived weighted probabilities of elastic and elastic–plastic response were used in modeling and simulating probabilistic behavior of von Mises and Drucker–Prager material models. Results show that the most likely response (mode) is different than the mean and/or deterministic solutions. In addition to that, smooth response curves (mode and mean) were observed for material models with linear hardening, even if deterministic response was characterized with a sudden change in stiffness.

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