Stochastic Elastic–Plastic Finite Elements

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Abstract

A computational framework has been developed for simulations of the behavior of solids and structures made of stochastic elastic–plastic materials. Uncertain elastic–plastic material properties are modeled as random fields, which appear as the coefficient term in the governing partial differential equation of mechanics. A spectral stochastic elastic–plastic finite element method with Fokker–Planck–Kolmogorov equation based probabilistic constitutive integrator is proposed for solution of this non–linear (elastic–plastic) partial differential equation with stochastic coefficient. To this end, the random field material properties are discretized, in both spatial and stochastic dimension, into finite numbers of independent basic random variables, using Karhunen–Loève expansion. Those random variables are then propagated through the elastic–plastic constitutive rate equation using Fokker–Planck-Kolmogorov equation approach, to obtain the evolutionary material properties, as the material plastifies. The unknown displacement (solution) random field is

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then assembled, as a function of known basic random variables with unknown deterministic coefficients, using polynomial chaos expansion. The unknown deterministic coefficients of polynomial chaos expansion are obtained, by minimizing the error of finite representation, by Galerkin technique.

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1. Introduction

In mechanics, simulations of static or dynamic behavior of solids and structures involve solutions of boundary value problems, which are comprised of the equilibrium equation, $A\sigma = \phi(t)$, together with the strain compatibility equation, $Bu = \epsilon$, and the constitutive equation, $\sigma = D\epsilon$, along with a set of additional restraints (boundary conditions). In the above, σ is generalized stress, $\phi(t)$ is generalized force that can be time (t) dependent, u is generalized displacements, ϵ is generalized strain and A, B, and D are operators which could be linear or non-linear.

Rigorous mathematical theory has been developed for problems where the only random parameter is the external force $\phi(t)$. In this case, the probability distribution function (PDF) of the response variable satisfies a Fokker–Planck-Kolmogorov (FPK) partial differential equation (cf. Soize [1]). With appropriate initial and boundary conditions the FPK PDE can be solved for

PDF of response variable. The numerical solution method of FPK equation corresponding to structural dynamics problems was described by number of researchers (e.g., Langtangen [2], Masud and Bergman [3]).

The other extreme case, which is of main interest of this paper, is when the stochasticity of the system is purely due to operator uncertainty. Exact solution of the problems with stochastic operator was attempted by Hopf [4], using characteristic functional approach. Later, Lee [5] applied the methodology to the problem of wave propagation in random elastic media and derived an FPK equation, satisfied by the characteristic functional of the random wave field. This characteristic functional approach is very complicated for linear problems and becomes even more intractable (and possibly unsolvable) for nonlinear problems with irregular geometries and boundary conditions.

Monte Carlo simulation technique is an alternative to analytical solution of partial differential equation with stochastic coefficient. A thorough review of different aspects of formulation of Monte Carlo technique for stochastic mechanics problem was presented in a state-of-the-art report edited by Schueller [6]. Monte Carlo technique has been widely used for probabilistic solution of uncertain boundary value problems (Paice et al. [7], Popescu et al. [8], Mellah et al. [9], DeLima et al. [10], Koutsourelakis et al. [11], Griffiths et al. [12], Nobahar [13], Fenton and Griffiths [14, 15]). It has the advantage that accurate solution can be obtained for any problem whose deterministic solution (either analytical or numerical) is known. However, the major disadvantage of Monte Carlo analysis is the repetitive use of the deterministic model until the solution variable becomes statistically significant. The computational cost associated with it could be very exorbitantly high (and probably intractable) for three-dimensional and/or non–linear problems with multiple uncertain material properties.

The difficulties with analytical solution and the high computational cost associated with Monte Carlo technique lead to the development of numerical methods for the solution of stochastic differential equation with random coefficient. For stochastic boundary value problems, stochastic finite element method (SFEM) is the most popular. There exist several formulations of SFEM, among which perturbation (Kleiber and Hien [16], Der Kiureghian and Ke [17]; Mellah et al. [9], Gutierrez and de Borst [18]) and spectral (Ghanem and Spanos [19], Keese and Matthies [20], Xiu and Karniadakis [21], Debusschere et al. [22], Anders and Hori [23]) methods are the most common. Nice reviews on the advantages and the disadvantages of different formulations of SFEM were provided by Matthies et al. [24] and recently, by Stefanou [25]. Mathematical issues regarding different formulations of SFEM were addressed by Deb et al. [26] and by Babuska and Chatzipantelidis [27]. Computational issues were discussed by Ghanem [28] and by Stefanou [25]. Recently, a hybrid treatment of various forms of epistemic uncertainties, including modeling error, was proposed by Soize and Ghanem [29]. However, most of the existing formulations are for linear elastic problems. Though there has been some published works on geometric non–linear problems (Liu and Der Kiureghian [30], Keese and Matthies [20], Keese [31]), there exist only few published papers on material non–linear (elastic–plastic) problems with uncertain material parameters.

The major difficulty in extending the available formulations of SFEM to general elastic–plastic problem is the high non–linear coupling in the elastic–

plastic constitutive rate equation. First attempt to propagate uncertainties through the elastic–plastic constitutive equation considering random Young's modulus was published only recently, by Anders and Hori [32, 23]. They took perturbation expansion at the stochastic mean behavior and considered only the first term of the expansion. In computing the mean behavior they took advantage of bounding media approximation. Although this method doesn't suffer from computational difficulty associated with Monte Carlo method for problems having no closed-form solution, it inherits "closure problem" and the "small coefficient of variation" requirements for the material parameters. Closure problem refers to the need for higher order statistical moments in order to calculate lower order statistical moments (cf. Kavvas [33]). The small COV requirement claims that the perturbation method can be used (with reasonable accuracy) for probabilistic simulations of solids and structures with uncertain properties only if their COVs are less than 20% (cf. Sudret and Der Kiureghian [34]). For soils and other natural materials, COVs are rarely below 20% (cf. Baecher and Christian [35]). Furthermore, with bounding media approximation, difficulty arises in computing the mean behavior when one considers uncertainties in internal variable(s) and/or direction(s) of evolution of internal variable(s). These difficulties, associated with probabilistic simulation of the elastic–plastic constitutive equation, prevent application of SFEM to general geomechanics problems.

Recently, a special nonlocal Eulerian-Lagrangian form of the Fokker-Planck-Kolmogorov (FPK) equation was derived by Kavvas [33] in order to model the probabilistic behavior of nonlinear/linear physical systems that have uncertain parameters, uncertain forcing functions and uncertain initial conditions, and that are described by conservation equations in the form of nonlinear/linear stochastic partial differential equations or stochastic ordinary differential equations. This Eulerian-Lagrangian form of the FPK was later applied by Jeremic et al. [36] to the probabilistic description of elasto-plasticity, by writing the generalized elastic-plastic constitutive rate equation in the probability density space. The non–linear (in real space) elastic–plastic constitutive rate equation becomes a linear partial differential equation in the probability density space, simplifying the solution process. The resulting FPK partial differential equation is second-order exact (analytical). Further, this FPK approach can very easily be specialized to any particular constitutive model. The probabilistic monotonic behavior of Drucker–Prager linear hardening and Cam Clay models were discussed by Sett et al. [37, 38]. Jeremić and Sett [39] later extended the FPK equation based *probabilistic elasto–plasticity* to include probabilistic yielding and in another paper Sett and Jeremić [40] discussed its influence on cyclic constitutive behavior of geomaterials. In the above publications, it was shown that with probabilistic approach to geomaterial modeling, even the simplest elastic–perfectly plastic model picks up some of the important features of geomaterial behavior, which deterministically, could only be possible with advanced constitutive models.

Encouraged by the probabilistic constitutive responses of geomaterials, in this paper, the authors, with the broad goal of understanding the influence of spatial uncertainties in probabilistic approach to geomaterial modeling, have extended the spectral formulation of stochastic finite element method (Ghanem and Spanos [19]) to general stochastic elastic–plastic problems.

Fokker–Planck–Kolmogorov equation based *probabilistic elasto–plasticity* has been used as probabilistic constitutive integrator in updating the material properties as the material plastifies. Both the formulation and numerical solution scheme are discussed.

2. FPK Equation based ***Probabilistic Elasto-Plasticity***

2.1. General Form of Constitutive Rate Equation

The constitutive behavior of many materials can be modeled by elastic–plastic constitutive rate equation, which, in most general form, can be written as:

$$\frac{d\sigma_{ij}}{dt} = D_{ijkl}\frac{d\epsilon_{kl}}{dt} \tag{1}$$

where ϵ is the strain, t is the pseudo time, and D is the modulus, which could be either elastic or elastic–plastic:

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{when elastic} \\[2ex] D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn}\dfrac{\partial U}{\partial \sigma_{mn}}\dfrac{\partial f}{\partial \sigma_{rs}}D^{el}_{rskl}}{\dfrac{\partial f}{\partial \sigma_{ab}}D^{el}_{abcd}\dfrac{\partial U}{\partial \sigma_{cd}} - \dfrac{\partial f}{\partial q_*}r_*} & \text{when elastic–plastic} \end{cases}$$

where, D^{el}, f, U, q_*, and r_* are elastic modulus, yield function, plastic potential function, internal variable(s), and rate(s) of evolution of internal variable(s) respectively. However, due to various uncertainties associated with our measurement of material properties, D in Eq. (1) becomes uncertain. This is especially significant for geomaterials, where coefficients of variation of measured properties are typically 50% or more. In traditional deterministic approach to material modeling, one typically applies engineering judgment (qualitative) in obtaining the 'most probable' material parameters and substitute in Eq. (1) to obtain the 'most probable' material constitutive behavior.

2.2. Constitutive Equation in Probability Density Space

In quantifying the uncertainties in soil constitutive behavior, Jeremić et al. [36] proposed a probabilistic approach to material modeling by writing the constitutive rate equation (Eq. (1)) in the probability density space using an approach based on Eulerian–Lagrangian form of Fokker–Planck–Kolmogorov equation (cf. Kavvas [33]). Eq. (1), when written in the probability density space, takes the form (Sett et al. [37]):

$$
\begin{aligned}
\frac{\partial P(\sigma_{ij}(t),t)}{\partial t} &= -\frac{\partial}{\partial \sigma_{mn}}\left[\left\{\left\langle \eta_{mn}(\sigma_{mn}(t), D_{mnrs}, \epsilon_{rs}(t))\right\rangle\right.\right.\\
&+ \int_0^t d\tau Cov_0\left[\frac{\partial \eta_{mn}(\sigma_{mn}(t), D_{mnrs}, \epsilon_{rs}(t))}{\partial \sigma_{ab}};\right.\\
&\left.\left.\left.\eta_{ab}(\sigma_{ab}(t-\tau), D_{abcd}, \epsilon_{cd}(t-\tau)\right]\right\} P(\sigma_{ij}(t),t)\right]\\
&+ \frac{\partial^2}{\partial \sigma_{mn}\partial \sigma_{ab}}\left[\int_0^t d\tau Cov_0\left[\eta_{mn}(\sigma_{mn}(t), D_{mnrs}, \epsilon_{rs}(t));\right.\right.\\
&\left.\left.\eta_{ab}(\sigma_{ab}(t-\tau), D_{abcd}, \epsilon_{cd}(t-\tau))\right] P(\sigma_{ij}(t),t)\right] \qquad (2)
\end{aligned}
$$

where, $P(\sigma(t), t)$ is the probability density of stress, $\langle\cdot\rangle$ is the expectation operator, and $Cov_0[\cdot]$ is the time-ordered covariance operator. One may also note that in Eq. (2), t is the pseudo-time of the constitutive rate equation (Eq. (1)) and η_{ij} is a random operator tensor, which is a function of stress tensor (σ_{ij}), material properties tensor (D_{ijkl}), and strain tensor (ϵ_{kl}). Eq. (2) can be written in a compact form as follows:

$$\frac{\partial P(\sigma_{ij},t)}{\partial t} = -\frac{\partial}{\partial \sigma_{mn}}\left[N_{(1)_{mn}} P(\sigma_{ij},t) - \frac{\partial}{\partial \sigma_{ab}}\left\{N_{(2)_{mnab}} P(\sigma_{ij},t)\right\}\right] \tag{3}$$

where, $N_{(1)}$ and $N_{(2)}$ are advection and diffusion coefficients respectively. With appropriate initial and boundary conditions, and given the second-order statistics of material properties, Eq. (3) can be solved for evolution of probability density of stress response with second-order accuracy, following any particular constitutive law. The advection and diffusion coefficients are function of statistical properties of material parameters and strain as well as the type of constitutive model. For example, following the general derivation in the Appendix of Jeremić et al. [36], it can be shown that the advection and diffusion coefficients for 1–D von Mises elastic-perfectly plastic shear stress (σ) versus shear strain (ϵ) constitutive relationship take the following form:

$$N_{(1)}^{vM} = \begin{cases} \frac{d\epsilon}{dt}\langle G\rangle & \text{when elastic} \\ 0 & \text{when perfectly plastic} \end{cases} \qquad N_{(2)}^{vM} = \begin{cases} t\left(\frac{d\epsilon}{dt}\right)^2 Var[G] & \text{when elastic} \\ 0 & \text{when perfectly plastic} \end{cases} \tag{4}$$

In Eq. (4), G is the elastic shear modulus, ϵ is the shear strain, $\langle\cdot\rangle$ is the expectation operator, and $Var\,[\cdot]$ is the variance operator. The transition from elastic to elastic–plastic can be controlled using mean yield criteria (Sett [41]) – shifting from elastic advection and diffusion coefficients to elastic–plastic advection and diffusion coefficients, when mean of elastic solution exceeds mean of yield stress. In mathematical terms, this translates to:

$$
\begin{array}{lll}
\text{if} & \langle f \rangle < 0 \vee (\langle f \rangle = 0 \wedge d\langle f \rangle < 0) & \text{then use elastic } N_{(1)} \text{ and } N_{(2)} \\
\text{or, if} & \langle f \rangle = 0 \vee d\langle f \rangle = 0 & \text{then use elastic–plastic } N_{(1)} \text{ and } N_{(2)}
\end{array}
\tag{5}
$$

2.3. Consideration for Probabilistic Yielding

If the yield strength of the material is uncertain, then there is always a possibility of "triggering" the elastic–plastic advection and diffusion coefficients in the pre–yield elastic region and vice versa. Mean yield criteria (Eq. (5)), however, fails to account for those possibilities. One possible solution could be to assign probability weights, based on cumulative density function (CDF) of yield strength (Σ_y), to the elastic and plastic advection and diffusion coefficients and solve the Fokker–Planck–Kolmogorov equation corresponding to that weighted average advection and diffusion coefficients to obtain full probabilistic elastic-plastic response. Mathematically, the equivalent (weighted) advection and diffusion coefficients ($N^{eq}_{(1)}$ and $N^{eq}_{(2)}$) can be written as (Jeremić and Sett [39]):

$$
\begin{aligned}
N^{eq}_{(1)}(\sigma) &= (1 - P[\Sigma_y \leq \sigma])N^{el}_{(1)} + P[\Sigma_y \leq \sigma]N^{pl}_{(1)} \\
N^{eq}_{(2)}(\sigma) &= (1 - P[\Sigma_y \leq \sigma])N^{el}_{(2)} + P[\Sigma_y \leq \sigma]N^{pl}_{(2)}
\end{aligned}
\tag{6}
$$

where $(1 - P[\Sigma_y \leq \sigma])$ represents the probability of material being elastic, while $P[\Sigma_y \leq \sigma]$ represents the probability of material being elastic–plastic. The superscripts $\cdot^{el}$ and $\cdot^{pl}$ on the advection and diffusion coefficients refer to pre–yield elastic region and post-yield elastic-plastic region. As an example, for von Mises elastic–perfectly plastic material (elastic and elastic–perfectly plastic advection and diffusion coefficients are given by Eq. (4)) with uncertain yield strength (Σ_y), the equivalent advection and diffusion coefficients

($N_{(1)}^{eq^{vM}}$ and $N_{(2)}^{eq^{vM}}$) would become:

$$\begin{aligned} N_{(1)}^{eq^{vM}}(\sigma) &= (1 - P[\Sigma_y \leq \sigma])\frac{d\epsilon}{dt}\langle G\rangle \\ N_{(2)}^{eq^{vM}}(\sigma) &= (1 - P[\Sigma_y \leq \sigma])t\left(\frac{d\epsilon}{dt}\right)^2 Var[G] \end{aligned} \tag{7}$$

2.4. Probabilistic Constitutive Simulation

Following the above described framework, any plasticity theory based constitutive model can be written in probability density space and the resulting FPK partial differential equation (PDE) can be solved, using appropriate initial and boundary conditions, to obtain the constitutive response probabilistically. In this paper, the FPK PDE is solved numerically using method of lines - by semidiscretizing the PDE in the stress domain using finite difference method and solving the resulting series of ordinary differential equations incrementally.

A typical, 1–D shear stress versus shear strain constitutive behavior of von Mises elastic–perfectly plastic material with uncertain shear modulus and uncertain yield (shear) strength is shown in Figure 1. In particular, Figure 1 shows the contour of evolutionary probability density function (PDF) of shear stress with shear strain, as the material is strained monotonically up to 1.026%. The evolutionary PDF was obtained as the solution of FPK equation (Eq. (3)) with equivalent von Mises elastic–perfectly plastic advection and diffusion coefficients (Eq. (7)). Shear modulus, with normal distribution, having a mean value of 61.4 MPa and a coefficient of variation of 38.5% was assumed in this simulation. Shear strength was also assumed to follow normal distribution with a mean of 0.2 MPa and a standard deviation of 0.14 MPa. Shear modulus and shear strength were assumed independent of each

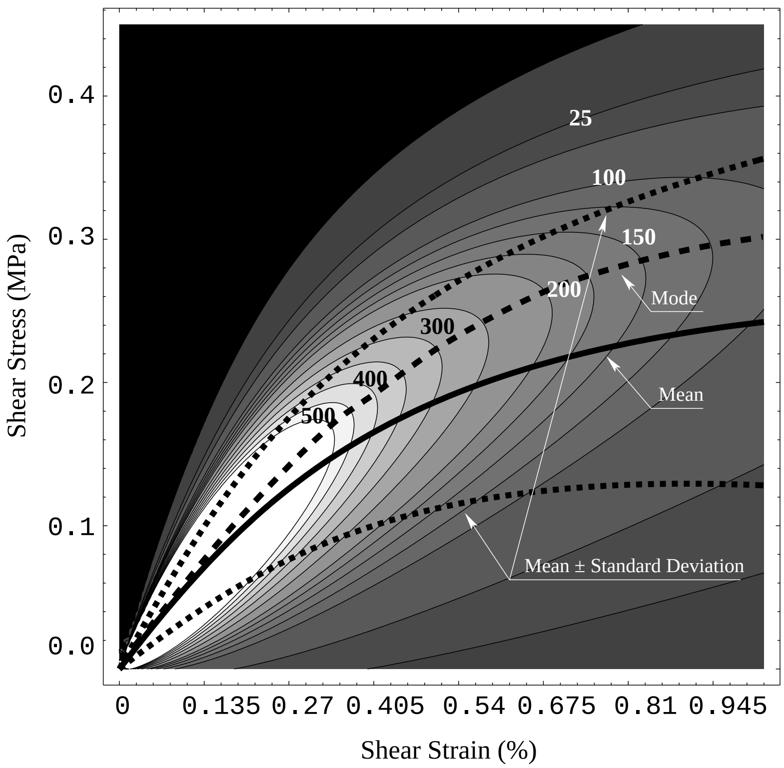
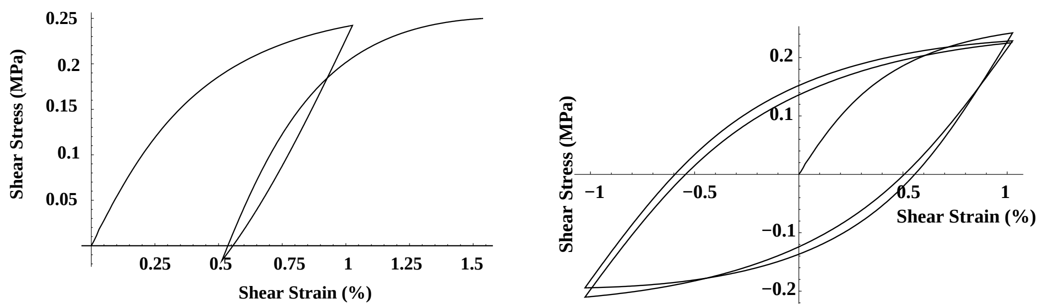


Figure 1: Contour of evolutionary probability density function (PDF) of shear stress, when von Mises elastic–perfectly plastic material with uncertain shear modulus and uncertain yield (shear) strength was monotonically strained; The mean, the mode, and the mean±standard deviation of shear stress – obtained by integrating the evolutionary PDF of shear stress – are also shown.

other. The assumed values are typical for clay material under undrained condition. In Figure 1, the mean and mean$\pm$standard deviation of shear stress, obtained by integrating the PDF of shear stress by standard techniques, are also shown. It is interesting to observe, in Figure 1, the non-Gaussian nature of the evolutionary PDF of shear stress. In other words, the mean shear stress differs from the most probable (mode) shear stress.

In addition to quantifying the uncertainties in predicted constitutive behavior, the probabilistic approach to elasto–plasticity could significantly influence material modeling in general. As can be observed from Figure 2, with probabilistic approach to material modeling, even the simplest elastic–perfectly plastic material model captures some of the features of (geo-)materials' behavior, for example, modulus reduction and modulus degradation as material undergoes cyclic loading. In particular, Figures 2(a) and (b) show the mean shear stress behaviors that are obtained from evolutionary PDF of shear stress when von Mises, elastic-perfectly plastic material (same as used for monotonic simulation) is strained cyclically. Figure 2(a) shows an unsymmetric loading–unloading–reloading cycle while Figure 2(b) shows a symmetric loading–unloading–reloading cycle. Deterministic simulation of modulus reduction and modulus degradation would require (far more) advanced constitutive models than the (probabilistic) elastic–perfectly plastic model used here. The influence of hardening rules on probabilistic approach to material modeling was discussed by Sett and Jeremić [40].



(a) (b)

Figure 2: Mean shear stress, obtained by integrating the evolutionary probability density function (solution of FPK equation (Eq. (3))), when von Mises elastic–perfectly plastic material with uncertain shear modulus and uncertain yield (shear) strength was cyclically strained: (a) loading–partial unloading–reloading cycle and (b) loading–complete unloading–reloading cycle; Two complete cycles are shown for the later

3. Formulation of Stochastic Elastic–Plastic Finite Element

3.1. Governing Equation

The governing partial differential equation in mechanics can be mathematically written, combining equilibrium, strain compatibility, and constitutive equations, as:

$$\Xi(\mathbf{x})u(\mathbf{x}) = \phi(\mathbf{x}) \tag{8}$$

where $\Xi(\mathbf{x})$ is a linear/non-linear differential operator, $\phi(\mathbf{x})$ is the external force and $u(\mathbf{x})$ is the response. If the material properties are uncertain (and modeled as random field), Ξ in Eq. (8) becomes a stochastic linear/non-linear differential operator and as a result, the response, u becomes a random field. Splitting the stochastic operator (Ξ) into a deterministic part (L) and a random part (Π), whose coefficients are zero-mean random fields, one can write Eq. (8) as:

$$[L(\mathbf{x}) + \Pi(\mathbf{x}, \theta)]\, u(\mathbf{x}, \theta) = \phi(\mathbf{x}) \tag{9}$$

In the above equation (Eq. (9)), θ is introduced to denote randomness in the variables. For example, the response $u(\mathbf{x}, \theta)$ is a spatial random function, where the probability distribution of u changes as a function of the location, $\mathbf{x}$, in the space continuum. Further, if one splits the input material properties random field into a deterministic trend and zero-mean random (uncertain) residual about trend, $\mathbb{D}(\mathbf{x}, \theta) = \hat{\mathbb{D}}(\mathbf{x}) + \mathbb{R}(\mathbf{x}, \theta)$, one can re-write Eq. (9) as:

$$\left[L_1(\mathbf{x})\hat{\mathbb{D}}(\mathbf{x}) + L_2(\mathbf{x})\mathbb{R}(\mathbf{x}, \theta)\right] u(\mathbf{x}, \theta) = \phi(\mathbf{x}) \tag{10}$$

where $L_1(\mathbf{x})$ and $L_2(\mathbf{x})$ are deterministic differential operators.

3.2. Spatial and Stochastic Discretization

Spectral approach to stochastic finite element formulation necessities discretization of Eq. (10) in both stochastic and spatial dimensions, as follows:

1. Karhunen–Loève (KL) expansion (Karhunen [42]; Loève [43]; Ghanem and Spanos [19]) is used to discretize the zero-mean fluctuating part of the input material properties random field ($\mathbb{R}(\mathbf{x}, \theta)$) into finite number of independent basic random variables,

$$\mathbb{R}(\mathbf{x}, \theta) = \sum_{n=1}^{L} \sqrt{\lambda_n} \xi_n(\theta) f_n(\mathbf{x}) \tag{11}$$

   where, λ_n and $f_n(\mathbf{x})$ are the eigenvalues and eigenvector, respectively of the covariance kernel of the zero-mean random field ($\mathbb{R}(\mathbf{x}, \theta)$). The zero-mean random variables, $\xi_n(\theta)$ are mutually independent and have unit variances.

2. Polynomial chaos (PC) expansion (Wiener [44]; Ghanem and Spanos [19]) is used to represent any unknown random variable in terms of known random variables. For example, any unknown random variable, $\chi(\theta)$ can be expanded, truncating after P terms, in functional (polynomial chaos, $\psi_i\left[\{\xi_r\}\right]$) of known random variables, ξ_r and unknown deterministic coefficients, γ_i, as:

$$\chi_j(\theta) = \sum_{i=0}^{P} \gamma_i^{(j)} \psi_i\left[\{\xi_r\}\right] \tag{12}$$

   Using PC expansion, the unknown partially-discretized – in spatial dimension, using KL expansion – displacement random field $(u(\mathbf{x}, \theta))$ in Eq. (10) can be further discretized in the stochastic dimension,

$$\begin{aligned} u(\mathbf{x}, \theta) &= \sum_{j=1}^{L} e_j \chi_j(\theta) b_j(\mathbf{x}) \\ &= \sum_{j=1}^{L} \sum_{i=0}^{P} \gamma_i^{(j)} \psi_i[\{\xi_r\}] c_j(\mathbf{x}) \\ &= \sum_{i=0}^{P} \psi_i[\{\xi_r\}] \sum_{j=1}^{L} \gamma_i^{(j)} c_j(\mathbf{x}) \\ &= \sum_{i=0}^{P} \psi_i[\{\xi_r\}] d_i(\mathbf{x}) \end{aligned} \tag{13}$$

   where $c_j(\mathbf{x}) = e_j b_j(\mathbf{x})$ and $d_i(\mathbf{x}) = \sum_{j=1}^{L} \gamma_i^{(j)} c_j(\mathbf{x})$

3. Shape function expansion (Zienkiewicz and Taylor [45], Bathe [46], Ghanem and Spanos [19]) is used to discretize the spatial component

– $d_i(\mathbf{x})$, in the above polynomial chaos expansion, Eq. (13) – of the unknown random field:

$$d_i(\mathbf{x}) = \sum_{n=1}^{N} d_{ni} l_n(\mathbf{x}) \tag{14}$$

where $l_m(\mathbf{x})$ are the shape functions.

3.3. Stochastic Finite Elements

Discretizing Eq. (10) in both spatial and stochastic dimensions using KL, PC, and shape function expansions, one can write Eq. (10) as,

$$\sum_{i=0}^{P} \sum_{n=1}^{N} \Bigg[\psi_i[\{\xi_r\}] L_1(\mathbf{x}) \hat{\mathbb{D}}(\mathbf{x}) l_n(\mathbf{x}) + \sum_{m=1}^{M} \xi_m(\theta) \psi_i[\{\xi_r\}] \; L_2(\mathbf{x}) \sqrt{\lambda}_m f_m(\mathbf{x}) l_n(\mathbf{x}) \Bigg] d_{ni} = \phi(\mathbf{x}) \tag{15}$$

Galerkin type procedure may be applied to Eq. (15) to solve for the unknown coefficients (d_{mi}) of the PC-expansion of the displacement random field and after some algebra, one may write Eq. (15) as (cf. Ghanem and Spanos [19]),

$$\sum_{n=1}^{N} K'_{mn} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} c_{ijk} K''_{mnk} = \Phi_m \left\langle \psi_i[\{\xi_r\}] \right\rangle \tag{16}$$

where, $\Phi_k = \int_D \phi(\mathbf{x}) l_k(\mathbf{x}) d\mathbf{x}$ and $c_{ijk} = \langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \rangle$. In the above equation (Eq. (16)), one may note that the expected values of the polynomial chaoses ($\psi_i[\{\xi_r\}]$) and the products of orthonormal random variables and polynomial chaoses ($\xi_k(\theta)\psi_i[\{\xi_r\}]\psi_j[\{\xi_r\}]$) can easily be pre-calculated in closed form (symbolically, for example using Mathematica [47]). Eq. (16) can be written in more familiar matrix form as,

$$\bar{K}\bar{u} = \bar{F} \tag{17}$$

where, $\bar{u}$ is the generalized displacement vector, $\bar{F}$ is the generalized force vector, and $\bar{K}$ is the generalized stiffness matrix, which is composed of two components, namely the deterministic stiffness matrix, K', defined as:

$$K'_{nk} = \int_D L_1(\mathbf{x}) l_n(\mathbf{x})\ \hat{\mathbb{D}}\ l_k(\mathbf{x}) d\mathbf{x} \tag{18}$$

and, the stochastic stiffness matrix, K'', defined as:

$$K''_{mnk} = \int_D L_2(\mathbf{x}) l_n(\mathbf{x}) \left\{ \sqrt{\lambda}_m f_m(\mathbf{x}) \right\} l_k(\mathbf{x}) d\mathbf{x} \tag{19}$$

3.4. Force-Residual Form

For non-linear problems (as attempted in this paper), Eq. (17) can be written in force-residual form and solved incrementally. One can write Eq. (17) in force-residual form as:

$$\bar{r}(\bar{u}, \lambda) = 0 \tag{20}$$

where, $\bar{u}$ are the generalized degrees of freedom and λ is the load control parameter. Differentiating Eq. (20), one can write the rate form of the force-residual equation (Eq. (20)) as,

$$\dot{\bar{r}} = \bar{K}\dot{\bar{u}} - \bar{q}\dot{\lambda} = 0 \tag{21}$$

where, $\bar{K}$ $(=\partial r_i/\partial u_j)$ is the tangent stiffness matrix and $\bar{q}$ $(= -\partial r_i/\partial \lambda)$ is the load vector. At regular points in the $\bar{u} - \lambda$ space, the tangent stiffness matrix is non–singular and hence one can solve the force-residual rate equation for $\dot{\bar{u}}$ as,

$$\dot{\bar{u}} = \left(\bar{K}^{-1}\bar{q}\right)\dot{\lambda} \quad \text{with} \quad \bar{u}^{'} = \frac{d\bar{u}}{d\lambda} = \bar{K}^{-1}\bar{q} \tag{22}$$

The resulting set of nonlinear ODEs (Eq. (22)) can be solved numerically for $\bar{u}$ with a set of initial conditions (at $\lambda = 0$, $\bar{u} = \bar{u}_0$) using either pure incremental method (forward Euler method) or incremental–iterative method (Newton method). This paper only considers the forward Euler method by which knowing the solution of $\bar{u}$ at the n^{th} step ($\bar{u}_n$), the solution at $(n+1)^{th}$ step ($\bar{u}_{n+1}$) can be obtained as (using load control),

$$\bar{u}_{n+1} = \bar{u}_n + \Delta\lambda_n \bar{u}_n^{'} \tag{23}$$

Hence, the incremental solution ($\Delta\bar{u}_n$) can be written as,

$$\Delta\bar{u}_n = \bar{u}_{n+1} - \bar{u}_n = \left(\bar{K}_n^{-1}\bar{q}_n\right)\Delta\lambda_n \tag{24}$$

3.5. Evaluation of Generalized Tangent Stiffness Matrix

In Eq. (24), the generalized tangent stiffness matrix ($\bar{K}$) needs to be re-evaluated at each step because the constitutive properties ($\hat{\mathbb{D}}$ in Eq. (18) and $\sqrt{\lambda_m}$ in Eq. (19)) of the material change as the material plastifies. The evolution of these material properties, as the material plastifies, are governed by the constitutive rate equation (Eq. (1)). At each integration point (Gauss point), the constitutive equation, written in probability density space (the FPKE, refer Eq. (3)), can be solved once for each KL-space[1] to obtain

[1]Note that KL-spaces are orthogonal to each other

updated $\hat{\mathbb{D}}$ (refer Eq. (18)) and $\sqrt{\lambda}_m$ (refer Eq. (19)), and thereby the generalized tangent stiffness matrix ($\bar{K}$; refer Eq. (24)) at each load step. For example, for 1–D problem, the (known, random) strain increment at the $(k-1)^{th}$ global step can be written as,

$$\Delta\epsilon^{k-1}(x,\theta) = \sum_{i=0}^{P} \psi_i[\{\xi_r\}](\theta) \sum_{n=1}^{N} \Delta d_{ni}^{k-1} B_n(x) \tag{25}$$

Hence, the advection and diffusion coefficients for the FPKE, assuming 1–D von Mises elastic–perfectly plastic material behavior, at the k^{th} global step can be calculated using Eq. (7) as follows:

For the zeroth KL-space (mean space):

$$\begin{aligned} N_{(1)}^{eq^{vM}k}(\sigma) &= (1/\Delta t)\ \hat{\mathbb{D}}\ \ (1 - P[\Sigma_y \leq \sigma]) \\ &\qquad \textstyle\sum_{i=0}^{P} \left\langle \psi_i[\{\xi_r\}] \right\rangle \sum_{n=1}^{N} \Delta d_{ni}^{k-1} B_n \\ N_{(2)}^{eq^{vM}k}(\sigma) &= t\ (1/\Delta t)^2\ \hat{\mathbb{D}}^2\ \ (1 - P[\Sigma_y \leq \sigma]) \\ &\qquad \textstyle\sum_{i=0}^{P} Var\left[\psi_i[\{\xi_r\}]\right] \left(\sum_{n=1}^{N} \Delta d_{ni}^{k-1} B_n\right)^2 \end{aligned} \tag{26}$$

For any other KL-spaces:

$$\begin{aligned} N_{(1)}^{eq^{vM}k}(\sigma) &= (1/\Delta t)\ \sqrt{\lambda}\ f\ \ (1 - P[\Sigma_y \leq \sigma]) \\ &\qquad \textstyle\sum_{i=0}^{P} \left\langle \xi\ \psi_i[\{\xi_r\}] \right\rangle \sum_{n=1}^{N} \Delta d_{ni}^{k-1} B_n \\ N_{(2)}^{eq^{vM}k}(\sigma) &= t\ (1/\Delta t)^2\ \lambda\ f^2\ \ (1 - P[\Sigma_y \leq \sigma]) \\ &\qquad \textstyle\sum_{i=0}^{P} Var\left[\xi\ \psi_i[\{\xi_r\}]\right] \left(\sum_{n=1}^{N} \Delta d_{ni}^{k-1} B_n\right)^2 \end{aligned} \tag{27}$$

In Eqs. (25)-(27), B is the derivative of shape function. One may note that the means and the variances of the polynomial chaoses ($\psi_i[\{\xi_r\}]$) and the means and the variances of the products of the orthonormal Gaussian random variables (ξ_m, where the subscript m denotes the KL-space) and the polynomial chaoses ($\psi_i[\{\xi_r\}]$) can easily be pre-calculated symbolically using Mathematica [47].

Assuming any value of pseudo-time increment Δt, the FPKEs corresponding to each KL-space (including the mean space) can be solved at $t = \Delta t$ to obtain the corresponding probability density function of stress for the strain increment given by Eq. (25). The explicit incremental solution scheme used here needs information on the new tangent material properties at each KL-space (for example, $\hat{\mathbb{D}}$ in Eq. (18) and $\sqrt{\lambda_m}$ in Eq. (19)) to increment forward. To this end, the constitutive rate equation (Eq. (1)), after taking expectation on both sides, can be written as (Kavvas [33]; Sett et al. [37]):

$$\frac{d\left\langle \sigma_{ij}(t)\right\rangle}{dt} = \left\langle D_{ijkl}(\sigma_{ij},t)\frac{d\epsilon_{kl}}{dt}\right\rangle + \int_0^t ds\ Cov_0\left[\frac{\partial}{\partial\sigma_{mn}}\{D_{ijkl}(\sigma_{ij(t)},t)\frac{d\epsilon_{kl}}{dt}\};D_{mnpq}(\sigma_{mn(t-s)},t-s)\frac{d\epsilon_{pq}}{dt}\right] \tag{28}$$

If one assumes $\sigma(t)$ to be a δ-correlated function of (pseudo) time, then the covariance term on the r.h.s of Eq. (28) becomes variance and hence, Eq. (28) can be re-written in incremental form as:

$$\frac{\Delta\left\langle \sigma_{ij}(t)\right\rangle}{\Delta t} = \left\langle D_{ijkl}(\sigma_{ij},t)\frac{\Delta\epsilon_{kl}}{\Delta t}\right\rangle + Var\left[D_{ijkl}(\sigma_{ij},t)\frac{\Delta\epsilon_{kl}}{\Delta t}\right] \tag{29}$$

It may be noted that if relatively large computational step size is used, then δ-correlation is a fairly good assumption for processes with finite correlation lengths as most of the information are in the variances of those processes. Eq. (29) relates the statistics of the tangent stiffness with the rate of change of mean stress, which is known, at each global load step increment, from the solution of FPKE. However, that itself doesn't allow for direct solution of the unknowns on the r.h.s., unless the l.h.s. (rate of change of mean stress) is known at several intermediate steps within the global load step increment. In such cases, the unknowns on the r.h.s. of Eq. (29) can be approximately solved by using a least square procedure e.g., Levenberg-Marquardt technique (Levenberg [48], Marquardt [49]). Hence, in this paper, at each Gauss integration point, the FPKE is solved in few sub-steps within each global step increment. By knowing the l.h.s. of Eq. (29) at those sub-steps and noting that the incremental strain is given by Eq. (25), the evolved tangent material properties of each space are approximately calculated. For example, for 1-D problem, for zeroth KL-space (mean space), Eq. (29) simplifies to:

$$\left.\frac{\Delta \langle \sigma(t) \rangle}{\Delta t}\right|_{zero^{th} KL-space} = \hat{\mathbb{D}}(t) \left\langle \frac{\Delta \epsilon}{\Delta t} \right\rangle + \hat{\mathbb{D}}^2(t) Var\left[\frac{\Delta \epsilon}{\Delta t}\right] \tag{30}$$

Now, if one assumes that, within the global load step, the mean tangent stiffness ($\hat{\mathbb{D}}(t)$) evolves quadratically [2] as $\hat{\mathbb{D}}(t) = a_1 + a_2 t^2$, where a_1 and a_2 are unknown deterministic coefficients, then one can write Eq. (30) as,

$$\begin{aligned} \left.\frac{\Delta \langle \sigma(t) \rangle}{\Delta t}\right|_{zero^{th} KL-space} = & \left(a_1 \left\langle \frac{\Delta \epsilon}{\Delta t} \right\rangle + a_1^2 Var\left[\frac{\Delta \epsilon}{\Delta t}\right]\right) + \\ & \left(a_2 \left\langle \frac{\Delta \epsilon}{\Delta t} \right\rangle + 2a_1 a_2 Var\left[\frac{\Delta \epsilon}{\Delta t}\right]\right) t^2 + \\ & a_2^2 Var\left[\frac{\Delta \epsilon}{\Delta t}\right] t^4 \end{aligned} \tag{31}$$

Using Eq. (25) and the same Δt that was used for solving the FPKE, the

[2] Linear evolution was also tried. However, quadratic evolution was preferred due to global step-size issue, especially at the transition region between elasticity and plasticity.

mean and variance terms on the r.h.s. of Eq. (31) can be easily evaluated and hence, knowing the l.h.s. of Eq. (31) over few sub-steps, the unknown deterministic coefficients (a_1 and a_2) can be determined using Levenberg-Marquardt technique. Hence, the evolved mean tangent material property ($\hat{\mathbb{D}}(t)$) at the end of global load step can be evaluated as $\hat{\mathbb{D}}(t) = a_1 + a_2\, t^2$ by substituting $t = \Delta t$.

Similarly, for any other KL-spaces, for 1-D problem, Eq. (29) simplifies to:

$$\left.\frac{\Delta \langle \sigma(t) \rangle}{\Delta t}\right|_{non-zeroKL-space} = \sqrt{\lambda}(t) f \left\langle \xi \frac{\Delta \epsilon}{\Delta t} \right\rangle + \lambda(t) f^2 Var\left[\xi \frac{\Delta \epsilon}{\Delta t}\right] \quad (32)$$

Again, within the global load step, one may assume that the tangent material property at the m^{th} KL-space ($\sqrt{\lambda_m}(t)$) evolves quadratically[3] as, $\sqrt{\lambda}(t) = a_3 + a_4\, t^2$, where a_3 and a_4 are unknown deterministic coefficients, and write Eq. (32) as,

$$\begin{aligned}\left.\frac{\Delta \langle \sigma(t) \rangle}{\Delta t}\right|_{non-zeroKL-space} = & \left(f a_3 \left\langle \xi \frac{\Delta \epsilon}{\Delta t} \right\rangle + f^2 a_3^2 Var\left[\xi \frac{\Delta \epsilon}{\Delta t}\right]\right) + \\ & \left(f a_4 \left\langle \xi \frac{\Delta \epsilon}{\Delta t} \right\rangle + 2 f^2 a_3 a_4 Var\left[\xi \frac{\Delta \epsilon}{\Delta t}\right]\right) t^2 + \\ & f^2 a_4^2 Var\left[\xi \frac{\Delta \epsilon}{\Delta t}\right] t^4 \end{aligned} \quad (33)$$

Hence, following the same procedure, as described above for the zeroth KL-space, the evolved tangent material property at the m^{th} KL-space ($\sqrt{\lambda_m}(t)$)

[3]As for the evolution of mean stiffness at the zeroth KL-space, here also, linear evolution was tried. However, quadratic evolution was preferred due to global step-size issue, especially at the transition region between elasticity and plasticity.

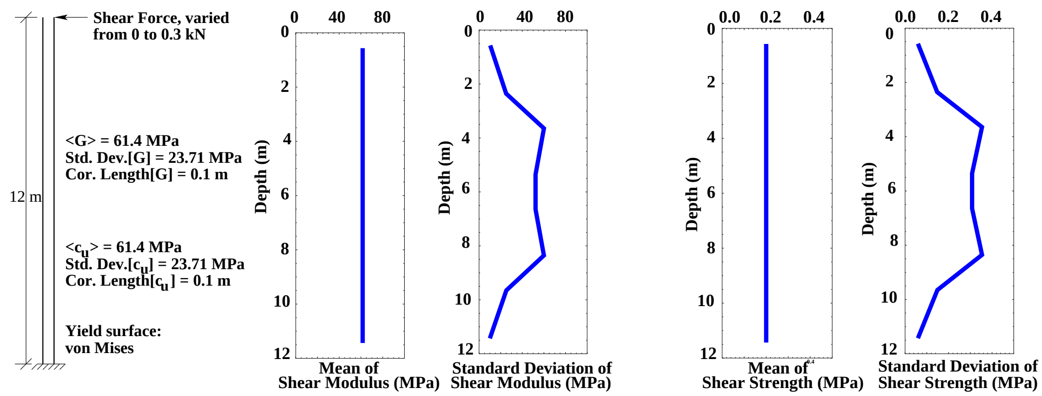
at the end of global load step can be estimated from Eq. (33) using Levenberg-Marquardt technique.

3.6. A Remark on Non-Gaussian Material Properties

The formulation presented above utilizes the properties of Gaussian distribution. However, the formulation can be generalized to include non-Gaussian (random field) material properties too. To this end, the usual technique of representing a non-Gaussian random variable in terms of Gaussian random variables through non-linear transformation, e.g., Rosenblatt transformation (Rosenblatt [50], Melchers [51], Grigoriu [52]) can be used. Alternately, as recently used by several researchers, classical polynomial chaos expansion (Ghanem [53], Sakamoto and Ghanem [54]) or generalized polynomial chaos expansion (Xiu and Karniadakis [55, 56], Foo et al. [57], Lucor et al. [58]) may also be utilized to represent non-Gaussian random fields. It may be noted that the FPK equation, used to integrate the constitutive equation at integration points, is general enough to treat both Gaussian and non-Gaussian random variables. See Sett and Jeremić [40], where FPK equation was applied to probabilistically simulate cyclic constitutive behavior of a von Mises material, whose shear strength was assumed to have a Weibull distribution.

4. SIMULATION RESULTS AND DISCUSSIONS

In this section, the spectral stochastic elastic-plastic finite element method, proposed in Section 3, is applied to simulate the probabilistic behavior of a 1–D soil column subjected to monotonic shear loading. The schematic of the soil column is shown in Figure 3(a). The shear modulus (G) of soil is assumed



(a) (b)

Figure 3: (a) Schematic of the 1-D stochastic soil shear beam; (b) Soil profile, in terms of means and standard deviations, of shear modulus and shear strength along the depth. Note that the standard deviation profiles are approximately calculated from the first two eigen modes of respective covariance kernels

to be a Gaussian random field with a mean of 61.4 MPa, a standard deviation of 23.7 MPa, and a correlation length of 0.1 m (covariance function is assumed to be exponential). The shear strength (c_u) of the soil is also assumed to be a Gaussian random field with a mean of 0.2 MPa, a standard deviation of 0.14 MPa, and a correlation length of 0.1 m (again, covariance function is assumed to be exponential). The shear modulus and shear strength are assumed independent of each other. Although the above material parameters are assumed, the statistical properties of the soil strength and stiffness parameters can be easily estimated by statistically analyzing commonly used in-situ test data e.g., standard penetration test (SPT) N-value or cone penetration test (CPT) q_T value (cf. Fenton [59, 60]). The soil profile, in terms of means and standard deviations of the shear modulus and shear strength, based on the above assumed statistical properties are shown in Figure 3(b). The standard deviations of shear modulus and shear strength, as shown in Figure 3(b), are approximately calculated at Gauss integration points using first two eigen modes of their respective covariance kernels. It is assumed that the soil follows von Mises yield criteria, which is a common assumption for simulation of undrained clay behavior. Further, it is assumed that after yielding the soil behaves as a perfectly plastic material. Typical 1-D shear stress versus shear strain constitutive behavior of von Mises elastic–perfectly plastic material under uncertainty were shown in Figures 1 – 2.

The input soil properties random fields are first discretized in both spatial and stochastic dimension using KL-expansion (Eq. (11)). To this end, the eigenvalues and eigenvectors of their respective covariance kernels are obtained numerically, using finite element technique. In this context, it may be noted that for exponential covariance kernel, as assumed in this example, the corresponding eigenvalue problem has a closed form solution and hence, a numerical treatment may be avoided. However, except for very few standard covariance kernels, numerical solution is necessary for any arbitrary covariance function. For details of finite element formulation of KL eigenvalue problem, the readers may refer to the book by Ghanem and Spanos [19]. In this study, open source FORTRAN library, LAPACK (Anderson et al [61]), is used for KL eigen analysis. For the stochastic simulation of the above soil column, only the first two eigenmodes for both shear modulus and shear strength are considered. In each loading step, the stochastic displacement (solution) of the soil column is assembled using PC-expansion (Eq. (12)). In this study, Hermite polynomial chaoses up to second order are considered. The unknown deterministic coefficients of the PC-expansion are then obtained by solving the spectral stochastic finite element system of equations (Eq. (16)). UMFPACK (Davis [62]), a set of routines for solving a sparse linear system using multifrontal method, is used for this purpose. At the constitutive level, FPKE (Eq. (3)) corresponding to the von Mises elastic perfectly plastic model (the advection and diffusion coefficients are given by Eqs. (26)-(27)) is solved at each KL space (including the mean space) to obtain the statistics of internal stresses as the soil evolves. To this end, the FPK partial differential equation is first semi-discretized in the stress domain on a uniform grid by central differences to obtain a series of ODE. These ODEs are then solved simultaneously, after incorporating boundary conditions, using a standard ODE solver SUNDIALS (Hindmarsh et al. [63]), which utilizes ADAMS method and functional iteration. Readily available open source code LMFIT (Moré et al. [64]) for Levenberg-Marquardt type least square regression analysis is used in estimating the evolution of statistical properties of material parameters from the evolutionary statistical properties of internal stress (refer to Eqs. (30)-(33)).

Figure 4 shows the evolutions of mean and mean±standard deviation of the displacement at the top of the soil column when a shear force, applied at the top of the soil column, is increased from 0 to 0.3 kN. It may be noted that, even though the material model is assumed elastic–perfectly plastic, due to the introduced uncertainties in yielding, the probabilistic response is non-linear from the beginning. Depending on the uncertainty in yield strength, there is always a possibility that the soil becomes elastic–plastic from the very beginning of loading. This possibility has been quantified, at

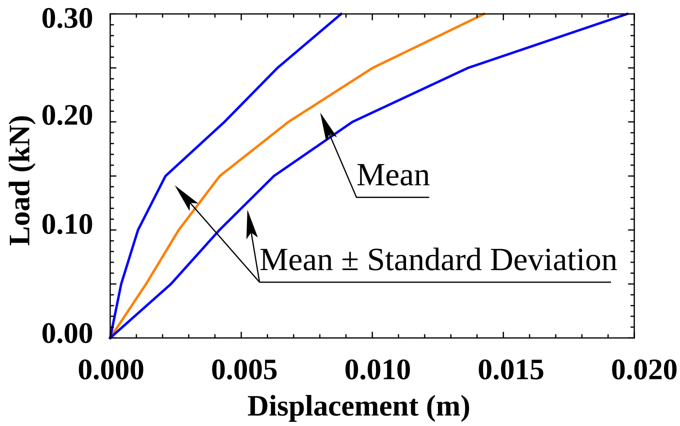


Figure 4: Evolution of mean and mean±standard deviation of displacement at the top of the 12 m tall stochastic soil column, when a deterministic static shear force is applied (varied from 0 to 0.3 kN) at top

the Gauss points, from the mean and variance of the yield strength of soil and taken into consideration implicitly during constitutive simulation using the equivalent advection and diffusion coefficients ($N_{(1)}^{eq^{vM}}$ and $N_{(2)}^{eq^{vM}}$, refer to Eq. (7)). These coefficients assign probability weights to the realizations of stress response based on the probability of material being elastic or elastic-plastic. As discussed in section 2.3, initially, at small strains, the probability of material being elastic-plastic is very small and hence, the initial probabilistic response (ensemble of all realizations) is closer (but not fully) to linear, elastic response. However, as deformation increases, the probability that the material yields increases and consequently, the probabilistic solution gradually becomes more elastic-plastic (Figure 4).

Uncertain tangent shear modulus (stiffness) evolves as well at each Gauss point – both in the mean and the probabilistic (KL) spaces – with the increase in shear loading. One such evolution (of mean of tangent shear modulus and of tangent of λ at the first and the second KL space), at a Gauss point located at a depth of 6.645 m from the top of the soil column, is shown in Figure 5.

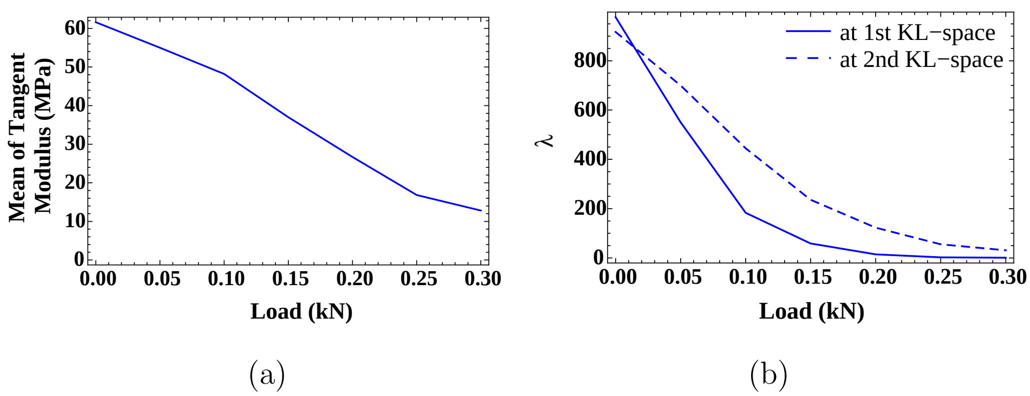
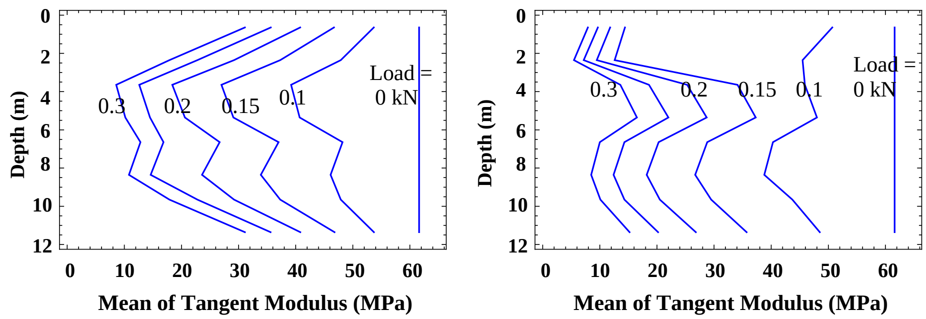


Figure 5: Evolution of the statistics of the tangent of the modulus at a depth of 6.645m from the top of the soil column, shear force at the top increases from 0 to 0.3 kN: (a) mean, (b) λ at the 1st and 2nd KL-space

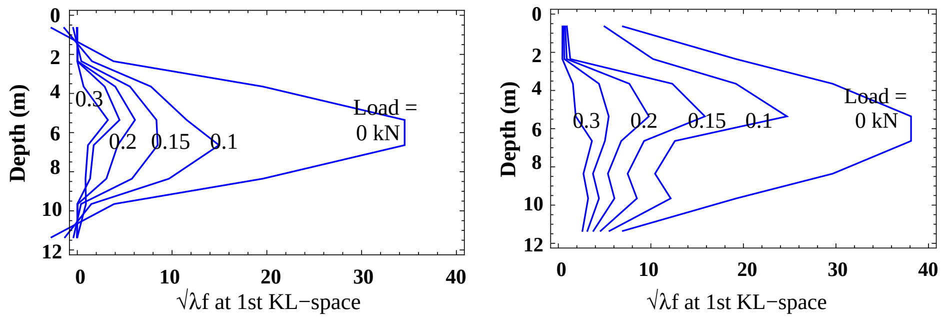
It is also interesting to observe the pattern of evolutionary material behavior along the depth of the soil column. The evolution of mean of tangent modulus along the depth of soil column is plotted in Figure 6(a). The mean of tangent modulus at each Gauss point evolves differently, depending on their respective advection and diffusion coefficient (refer to Eq. (26)). As can be observed from Figure 6(a), the material around the middle of the soil column has evolved more than that around the top and the bottom of the soil column. In other words, middle of the soil column has gotten softer than the top and bottom. It may, however, be noted that this stiffness reduction pattern depends on the correlation lengths of the random field material properties. For example, Figure 6(b) shows the stiffness reduction pattern of the same soil column, but with a shear modulus correlation length of 1 m.



(a) (b)

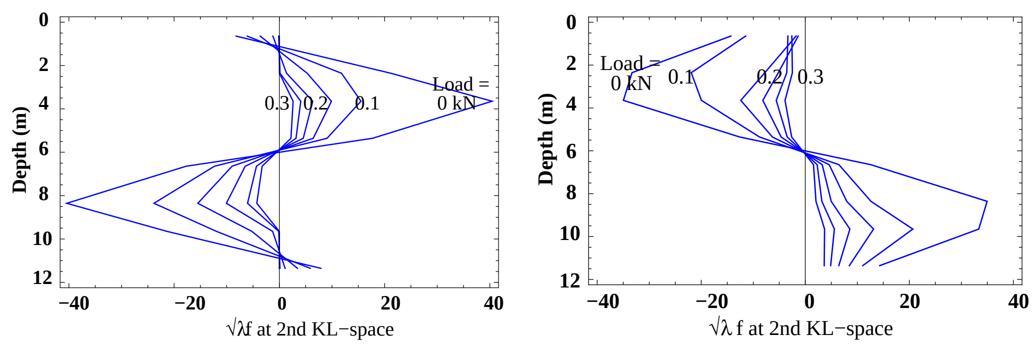
Figure 6: Evolution of mean of tangent modulus along the depth of soil column with load: (a) shear modulus correlation length 0.1m, and (b) shear modulus correlation length 1.0m (all other parameters are kept the same).

The variabilities in the above evolved mean behaviors are captured in terms of $\sqrt{\lambda} f$, which is a measure of standard deviation, at the mutually perpendicular KL spaces. Figures 7(a) and (b) show the evolutions of $\sqrt{\lambda} f$ of the tangent modulus at the 1st KL-space for the soil column with correlation length of shear modulus of 0.1m and 1.0 m, respectively. Figures 8(a) and (b) show the same for the 2nd KL-space. As can be observed from Figures 7 and 8 that at each Gauss point, as in the mean space, the material properties at the KL-spaces also evolved in different fashions, depending upon their respective advection and diffusion coefficients (refer to Eq. (27)). The evolution of standard deviation of the tangent modulus, which is the absolute value of the sum of evolutionary $\sqrt{\lambda} f$ over all KL spaces, is shown in Figure 9. Comparing Figures 6 and 9, it might be concluded that the pattern of stiffness reduction along the depth of the soil column and its associated uncertainty depend on the correlation length of the soil modulus.



(a) (b)

Figure 7: Evolution of tangent $\sqrt{\lambda_1}\ f_1$ along the depth of soil column with load: (a) shear modulus correlation length 0.1m, and (b) shear modulus correlation length 1.0m (all other parameters are kept the same).



(a) (b)

Figure 8: Evolution of tangent $\sqrt{\lambda_2}\ f_2$ along the depth of soil column with load: (a) shear modulus correlation length 0.1m, and (b) shear modulus correlation length 1.0m (all other parameters are kept the same).

The influences of the variances of shear modulus and shear strength on the displacement behavior at the top of the soil column have also been studied.

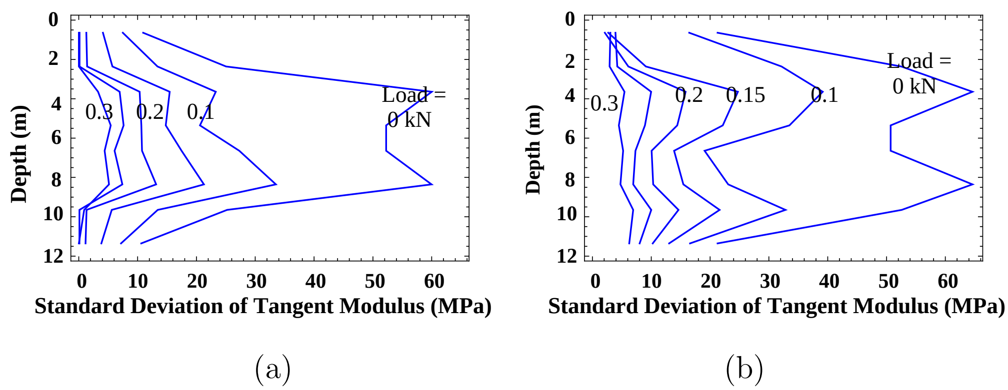


Figure 9: Evolution of standard deviation of tangent modulus along the depth of soil column with load: (a) shear modulus correlation length 0.1m, and (b) shear modulus correlation length 1.0m (all other parameters are kept the same).

Figure 10(a) shows the mean and mean±standard deviation of displacement at the top of the soil column when COV of shear strength is assumed to be 5%, keeping everything else the same as the original model. As expected, less non-linear response is obtained in the domain of the simulation (compare with Fig. 4). Low uncertainty (COV = 5%) in shear modulus, on the other hand, yields a more non-linear response (Figure 10(b)) than the case where the shear strength is less uncertain (Figure 10(a)), but less non-linear response than the original model where both the shear strength and the shear modulus are very uncertain (Figure 4).

5. CONCLUDING REMARKS

A computational formulation is developed for numerical solution of elastic–plastic boundary value problems with stochastic coefficients. The framework is based on spectral formulation of stochastic finite element method. At the

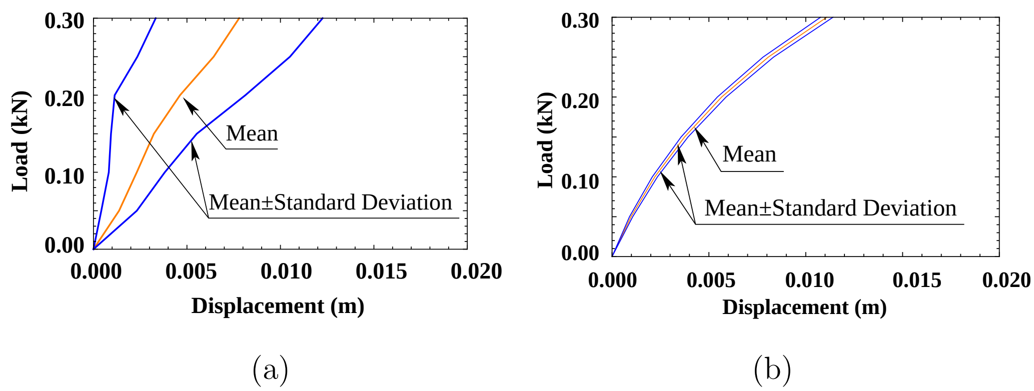


Figure 10: Evolution of mean and mean±standard deviation of displacement at the top of the 12 m tall stochastic soil column, assuming (a) low uncertainty (COV = 5%) in shear strength and (b) low uncertainty (COV = 5%) in shear modulus. All other material parameters are assumed to be the same as the original model, where both the shear strength and the shear modulus are very uncertain

Gauss integration points, Fokker-Plank-Kolmogorov equation based probabilistic constitutive integrator is used to update – as the material plastifies – the statistical properties of the tangent modulus. The advantage of Fokker-Plank-Kolmogorov equation based probabilistic elasto–plasticity is that it transforms the nonlinear stochastic elastic-plastic constitutive rate equation in the real space into a linear deterministic partial differential equation in the probability density space. This deterministic linearity simplifies the numerical solution process of the probabilistic constitutive equations. In addition, Fokker-Plank-Kolmogorov approach to probabilistic elasto-plasticity yields second-order accurate (analytical) probabilistic constitutive solution and doesn't suffer from 'closure problem' and 'small variance requirement', associated with perturbation technique and high computation cost, associated with Monte Carlo technique. Further, the presented formulation is general enough to be used with any elastic-plastic constitutive models with uncertain material parameters. Different constitutive models will only yield different advection and diffusion coefficients for the constitutive Fokker-Planck-Kolmogorov equation.

Developed stochastic framework is applied in simulating the behavior of a 1-D stochastic soil shear column subjected to a deterministic shear force at the top. The solution has been presented in terms of mean and standard deviation of displacement. The evolutionary pattern of stiffness reduction along the depth of the soil column is discussed. The influences of the correlation length and the variance of the soil parameters on the response have also been addressed.

On a closing note, while, in this paper, the application examples for the developed framework deal with the uncertain soil, the formulation is generic and can be applied to any elastic-plastic or elastic-damage material with uncertain material properties.

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