FINITE DEFORMATION ANALYSIS
OF GEOMATERIALS

BY

B. JEREMIC
K. RUNESSON
S. STURE

DEPARTMENT OF CIVIL & ENVIRONMENTAL ENGINEERING
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA AT DAVIS

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Abstract
The early extensions to large deformation of finite strain methods involve the use of the large deformation strain, which is an approximation to the small deformation strain. The error in using the small deformation strain measure instead of the large deformation strain is significant. The error is approximately 10% after a certain level of deformation. Only very small deformations can approximate contact with the large deformation strain.

\[
\frac{(\varepsilon_2^r n + \varepsilon_1^r n)^2}{1} = \frac{(\varepsilon_2 n + \varepsilon_1 n)^2}{1} = \varepsilon_r^l
\]

Clearly, the difference between \( \varepsilon^r \) and \( \varepsilon^s \) is in the nonlinear terms of the strain tensor, \( \varepsilon_{ij}^r \) and \( \varepsilon_{ij}^s \). To this end, we use the definition of a deformation gradient tensor, the response of a solid in terms of small and large deformation problems. The difference between large and small deformation solutions is presented, with distinctions between large and small deformation equations. A simple example is presented, with the results of large and small deformation solutions.

It is important to note that strains are non-linear functions of displacement fields and thus

The interaction schemes, the pioneering work of Shin and Taylor,\textsuperscript{48} and Shin and Guruswamy and de Ruiter,\textsuperscript{49} for most of the theories, such as the linear approach and the linearized strain tensor. The linearized strain tensor has been derived (slanting with a few assumptions). The large deformation interaction schemes have in recent years been proven to be accurate. The large deformation interaction schemes from Taylor and Shin,\textsuperscript{48} and from Shin and Shin,\textsuperscript{49} have been developed in recent years. The large deformation interaction schemes from Shin and Shin,\textsuperscript{49} have been developed in recent years.

The key assumption in implementing deformation elastoplasticity in the additive decomposition of strain into elastic and plastic components holds only for a number of generalized models. The models in the interaction schemes are designed to capture the overall behavior of the material and the interaction between the elastic and plastic components. The models are designed to capture the overall behavior of the material and the interaction between the elastic and plastic components.
plastic analysis of solids was conducted in the Lagrangian form\(^1\). Large deformation principle of virtual work based formulation for large strain elastic–plastic analysis of solids in the Lagrangian form was proposed by Hibbitt et al. [16]. The Eulerian form of the solution to the problem was proposed by McMeeking and Rice [32]. The disadvantage with this approach was in the necessary use of incrementally objective integration algorithms that may be computationally expensive. Hypoelastic based techniques, aimed at problems with small elastic strains were also proposed by many others, (see for example Saran and Runesson [42]). A number of problems encountered with different stress rates were noted by Nagtegaal and de Jong [34], Kojić and Bathe [21] and Szabó and Balla [52].

On the other hand, hyperelastic based techniques have been developed recently for purely deviatoric plasticity, for example by Simo and Ortiz [47], [45], Bathe et al. [2], Simo [43], [44], Eterovic and Bathe [12], Perić et al. [39] and Cuitino and Ortiz [10]. Most of the multiplicative split techniques are based on the earlier works of Hill [17], Bilby et al. [3] Kröner [23], Lee and Liu [26], Fox [14] and Lee [25].

Simo and Ortiz [47] where the first to propose a computational approach entirely based on the multiplicative decomposition of the deformation gradient. Their stress update algorithm, however, used the cutting plane scheme that has been shown by de Borst and Feenstra [11] to yield erroneous results for some yield criteria. Bathe et al. [2] have used the multiplicative decomposition with logarithmic stored energy function and an exponential approximation of the flow rule for non–linear analysis of metals. Eterovic and Bathe [12] included kinematic

\(^1\)Hypoelasticity is presented in spatial format. Virtual work is normally stated in the material format.
tensors, which renders them unsuitable for anisotropic hardening/softening material models. The above develop-
ments make an implicit assumption of co-geometry of principal directions of stress and strain. Nonlinear elastic laws for the analysis of time- and space-scale effects of 
strain cannot bear on the problems of the present study. The analysis was conducted for a linear hardening law, and the 
material model is regarded as a strain-space. They followed a method which has been shown by 
Strain: The applied model, they followed the strain-strain curve. Although 
small strain state update algorithms and their corresponding consistent tangent stiffness 
matrix were made for the linear deformation regime. However, developments were made for 
the finite deformation regime. They have also exploited the use of a series of 
approximations. They also restricted the use of their algorithms to the small strain 
approximation. There are also different kinds of nonlocal theories and their 
development is obtained only.
(6) \[ 0 = (\eta \bar{q})_{\text{ext}} + (\eta \bar{n} \bar{m} \bar{n})_{\text{int}} \]

The virtual work equation (4) can be written as:

(7) \[ (\eta \bar{n} \bar{m} \bar{n})_{\text{int}} \eta = \frac{\tau \bar{m} + \tau \bar{n}}{1} + (\tau \bar{m} + \tau \bar{n}) \frac{\tau \bar{m} + \tau \bar{n}}{1} = (\eta \bar{n} \bar{m} \bar{n})_{\text{int}} \]

where we have used the symmetry of \( \eta \) and definition for deformation gradient.

(4) \[ \lambda \eta \left( \int (\tau \bar{n} \eta \bar{m} + \tau \bar{m} \eta \bar{n}) + (\tau \eta \bar{m} + \tau \bar{n} \eta \bar{m}) \right) \frac{\partial}{\partial \eta} \int = \lambda \eta \int \lambda \bar{q} \eta \partial \int \]

Kirchhoff stress tensor \( \sigma \). The proof is then to rewrite the left hand side of (3) by using the symmetric second Piola–Kirchhoff stress tensor.

(3) \[ S \bar{p} \frac{\partial}{\partial \eta} \int - \lambda \int \lambda \bar{q} \eta \partial \int = \lambda \int \lambda \bar{q} \eta \partial \int \]

The integral formulation of (0) initial volume \( V_0 \), with virtual displacements \( \eta \) and \( \bar{m} \) and \( \bar{n} \) are body forces. The weak form of the equilibrium equation is obtained by integrating by parts with reference to the initial configuration and into equilibrium equations.

(2) \[ 0 = \eta \partial \bar{p} - \tau \bar{m} \]

The local form of equilibrium equations in Lagrangian format for the static case can be written as:

Formulation

2 Material and Geometric Non-linear Finite Element
\[
\begin{align*}
\text{(13)} \quad & \mathcal{A} \left( h^2 t^2 n^2 V + t^2 n^2 v^2 n + \left( f^2 t^2 n + f^2 n^2 \right) \right) \sum_{I} \int \frac{V}{I} \left( \sum_{I} \nabla n_{\text{in}} \nabla ng \right) M_{11} V \\
\text{or, by conventionally splitting the above equation we can write} \quad & \\
\text{(12)} \quad & \mathcal{A} \left( \left( f^2 t^2 n^2 V + \frac{t^2 n^2 v^2 n}{I} \right) \sum_{I} \int \frac{V}{I} \right) + \\
& + \mathcal{A} \left( \left( f^2 t^2 n^2 V + \frac{t^2 n^2 v^2 n}{I} \right) \sum_{I} \int \frac{V}{I} \right) = \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V
\end{align*}
\]

To this end, (11) can be rewritten by expanding the definition for \( \nabla \) as

\[
\text{(11)} \quad \mathcal{A} \left( \left( f^2 t^2 n^2 V + \frac{t^2 n^2 v^2 n}{I} \right) \sum_{I} \int \frac{V}{I} \right) = \mathcal{A} \left( \left( f^2 t^2 n^2 V + \frac{t^2 n^2 v^2 n}{I} \right) \sum_{I} \int \frac{V}{I} \right)
\]

Here we have used \( \mathcal{A} \left( \left( f^2 t^2 n^2 V + \frac{t^2 n^2 v^2 n}{I} \right) \sum_{I} \int \frac{V}{I} \right) = \mathcal{A} \left( \left( f^2 t^2 n^2 V + \frac{t^2 n^2 v^2 n}{I} \right) \sum_{I} \int \frac{V}{I} \right)
\]

where

\[
\text{(10)} \quad \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V = \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V
\]

\[
\text{(9)} \quad \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V + \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V = \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V
\]

\[
\text{(8)} \quad S P \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V - \mathcal{A} \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V = \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V
\]

\[
\text{(7)} \quad \mathcal{A} \left( \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V = \left( \sum_{I} \frac{t^2 n^2 \nabla \nabla ng}{M_{11}} \right) M_{11} V
\]

where
\[ h^n H = \gamma_n \approx i_n \]

Standard finite element discretization of the displacement field is:

\[ S p^{\mathbf{f}, n} q^{\mathbf{g}} \int - A p^{\mathbf{f}, n} q^{\mathbf{g}} \Psi^{\mathbf{h}} \int = (\mathbf{n}^1 g)_{x \cdot \mathbf{A} \cdot \mathbf{M}} \]

\[ A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int + A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int = (\mathbf{n}^1 g^0 u^i n^{j g})_{\mathbf{m} \cdot \mathbf{M}} \]

The weak form of equilibrium expressions for internal and external virtual work, with the above mentioned symmetry of \( S \) can be written as

\[ A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int = (\mathbf{n}^1 g^1 n^{j n^{k g}}) A_k \nabla \]

with

\[ A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int + A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int + \]

\[ A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int + \]

\[ A p^{\mathbf{f}, \mathbf{S}} f^{\mathbf{n}, m} q^{\mathbf{g}} \int = (\mathbf{n}^1 g^1 n^{k g}) A_k \nabla \]

Moreover, it can be shown (e.g. [61]) that an additional symmetry induced by the non-associated flow rule and the loss of major symmetry in the second Piola-Kirchhoff stress tensor reduces the loss of major symmetry, which is induced by the non-associated flow rule. However, major symmetry cannot be guarantied.

It should be noted that the algebraic tangent stiffness (ATLS) tensor possesses the form:

\[ \mathbf{A} \mathbf{p}^{\mathbf{f}, \mathbf{S}} (f^{\mathbf{n}, m} q^{\mathbf{g}} + f^{\mathbf{n}, g} q^{\mathbf{f}}) \mathbf{A} = (\mathbf{n}^1 g^1 n^{k g}) A_k \nabla \]

By further rearranging and collecting terms we can write:

\[ \mathbf{A} \mathbf{p}^{\mathbf{f}, \mathbf{S}} (f^{\mathbf{n}, m} q^{\mathbf{g}} + f^{\mathbf{n}, g} q^{\mathbf{f}}) \mathbf{A} = (\mathbf{n}^1 g^1 n^{k g}) A_k \nabla \]
\[ \lambda \left( \nabla \right)^2 + \lambda \left( \nabla \right)^2 + \lambda \left( \nabla \right)^2 = \lambda \left( \nabla \right)^2 \]

The global affine tangent stiffness matrix (tensor) is given as:

\[ SP \left( iH \right)^{ij} + \lambda \left( \nabla \right)^2 \]
that by performing the integrations in the intermediate configuration, we obtain the
expression

\[
\begin{align*}
\mathbf{\lambda}^{\text{int}} &= \left( \mathbf{\lambda}^{\text{int}} \right)^\text{int} + \mathbf{\lambda}^{\text{int}} - \mathbf{\lambda}^{\text{int}} \\mathbf{\lambda}^{\text{int}} \\
\mathbf{\sigma}^{\text{int}} &= \left( \mathbf{\sigma}^{\text{int}} \right)^\text{int} + \mathbf{\sigma}^{\text{int}} - \mathbf{\sigma}^{\text{int}} \mathbf{\sigma}^{\text{int}} \\
\mathbf{\nu}^{\text{int}} &= \left( \mathbf{\nu}^{\text{int}} \right)^\text{int} + \mathbf{\nu}^{\text{int}} - \mathbf{\nu}^{\text{int}} \mathbf{\nu}^{\text{int}}
\end{align*}
\]

where

\[
\mathbf{\lambda}^{\text{int}} = \left( \mathbf{\lambda}^{\text{int}} \right)^\text{int} + \mathbf{\lambda}^{\text{int}} - \mathbf{\lambda}^{\text{int}} \mathbf{\lambda}^{\text{int}}
\]

obtained from the equation

\[
\mathbf{\nu}^{\text{int}} = \left( \mathbf{\nu}^{\text{int}} \right)^\text{int} + \mathbf{\nu}^{\text{int}} - \mathbf{\nu}^{\text{int}} \mathbf{\nu}^{\text{int}}
\]

and

\[
\mathbf{\sigma}^{\text{int}} = \left( \mathbf{\sigma}^{\text{int}} \right)^\text{int} + \mathbf{\sigma}^{\text{int}} - \mathbf{\sigma}^{\text{int}} \mathbf{\sigma}^{\text{int}}
\]

The vector of externally applied forces is given as

\[
\mathbf{\mathbf{f}} = \mathbf{\mathbf{f}}^{\text{int}} + \mathbf{\mathbf{f}}^{\text{ext}}
\]

while the load vector from internal stresses is given as

\[
\mathbf{\mathbf{f}}^{\text{int}} = \int \mathbf{\mathbf{f}}^{\text{int}} \mathbf{\mathbf{g}}^{\text{int}}
\]

The vector of externally applied loads is then

\[
\mathbf{\mathbf{f}}^{\text{ext}} = \int \mathbf{\mathbf{f}}^{\text{ext}} \mathbf{\mathbf{g}}^{\text{ext}}
\]

The global equilibrium tangent stiffness matrix contains both the linear strain incremental

\[
\mathbf{\mathbf{K}}^{\text{int}} = \frac{\partial \mathbf{\mathbf{f}}^{\text{int}}}{\partial \mathbf{\mathbf{u}}^{\text{int}}}
\]

and

\[
\mathbf{\mathbf{K}}^{\text{ext}} = \frac{\partial \mathbf{\mathbf{f}}^{\text{ext}}}{\partial \mathbf{\mathbf{u}}^{\text{ext}}}
\]
By using the axioms of material frame indifference, we conclude that $M$ depends only on

$\nabla H$ and $G$, that is:

$$ (XY, H, G) M = M $$

Thus, the generalized form of the elastic potential function is described in equation (39), with restriction to the most general form of the elastic potential function, the corresponding stress component. The most simple component is determined by the strain energy function and volume of the undeformed configuration. A material is called hyperelastic or Green elastic if there exists an elastic potential function

### 3.1 Hyperelasticity

#### 3 Finite Deformation Hyperelastic–Plasticity

Some isotropic materials, in the solids, both isotropic and anisotropic. This generalization will be further enhanced in section 4. The formulation presented above is rather general and relevant to a large set of materials.

$$ \begin{array}{l}
\sigma_{ij} x_i = \int_{1}^{u} \frac{\partial f}{\partial x_i} \ dt + u \frac{\partial f}{\partial \alpha} \\
\sigma_{ij} x_i = \int_{1}^{u} \frac{\partial f}{\partial x_i} \ dt + u \frac{\partial f}{\partial \alpha}
\end{array} $$

The immediate contribution to the initial state stress and $\sigma_{ij}$ tensor in the initial configuration we need to perform a pull-back from tensor $\sigma_{ij}$ and then obtained based on $\sigma_{ij}$. In order to obtain the second phase–Kuhn–Munkers stress and subsequently the second phase–Kuhn–Munkers stress

$$ \frac{\partial f_{ij}}{\partial \alpha} = \frac{\partial f_{ij}}{\partial \alpha} $$

See Alexander and Higgen [191, pp. 317]. See Alexander and Higgen [191, pp. 317].
In order to follow the consistency of index notation in this work, we shall make an effort to represent all the essential quantities in indicial form.

In actual equation (14), we can also be written as
\[ V \left( P, \frac{f_{N(v)}^N}{} \right) \frac{f_{N(v)}^N}{} = \frac{f_{N(v)}^N}{} \]

where
\[ V \left( P, \frac{f_{N(v)}^N}{} \right) \frac{f_{N(v)}^N}{} = \frac{f_{N(v)}^N}{} \]

and the present case is the $\phi$-in equation with members in parentheses. For example, in

\[ 0 = \varepsilon I + \nu \varepsilon I - \nu \varepsilon I + \nu \nu I - \int_{\phi\phi} \left( \nu \nu \right) d \]

then calculate (9) of the characteristic polynomial.

\[ \begin{align*}
I & = \frac{f_{N(v)}^N}{} \\
(\frac{f_{N(v)}^N}{} \frac{f_{N(v)}^N}{} - \frac{f_{N(v)}^N}{} I) & = \frac{f_{N(v)}^N}{} I
\end{align*}
\]

where

\[ \begin{align*}
(\varepsilon I, \varepsilon I, \varepsilon I) M_1 & = (\varepsilon \nu \varepsilon I, \varepsilon \nu \varepsilon I, \varepsilon \nu \varepsilon I) M_1 = M_1
\end{align*}
\]

\[ (P, N, X) M = (N, X, M) M = (M, X, N) M \]

\[ M = \left( \phi \phi \right) \frac{f_{N(v)}^N}{} \left( \frac{f_{N(v)}^N}{} \phi \phi \right) M = \left( \phi \phi \right) \frac{f_{N(v)}^N}{} \left( \frac{f_{N(v)}^N}{} \phi \phi \right) M \]

The right and left stretch tensors, when and left stretch tensors, have the same principal values (principal stretches).

\[ \text{Rigid and left stretch tensors, \textit{see also \textit{Mandel \textit{Collins-Gray-Mandel}}}.
\]

\[ \text{In the case of material isometry, the strain energy function } M \text{ belongs to the class of isotropic invariant scalar functions. It satisfies the relation:}
\]
The most general form of the isotropic strain energy function \( W \) in terms of principal stretches can be expressed as:

\[
W = W(\epsilon_1, \epsilon_2, \epsilon_3, \gamma_1, \gamma_2, \gamma_3)
\]

Similarly we can obtain:

\[
W = W(\epsilon_1, \epsilon_2, \epsilon_3, \gamma_1, \gamma_2, \gamma_3)
\]

where indices \( \nu, \rho \) are cycle permutations of \( 1, 2, 3 \). It follows directly from the definition:

\[
W = W(\epsilon_1, \epsilon_2, \epsilon_3, \gamma_1, \gamma_2, \gamma_3)
\]

It should be noted that the denominator in equation (46) can be written as:

\[
\epsilon_1^e I + \epsilon_2^e I - \Gamma \nabla - \Gamma \nabla - \Gamma \nabla - \Gamma \nabla
\]

Then, the Lagrangian strain tensor algebra by the Lagrangian strain tensor and the Lagrangian strain tensor have used previously in order to develop a useful representation for generalized strain tensors throughout \( \Gamma \). After some analysis, the following theorem in order to develop a useful representation for generalized strain tensors throughout \( \Gamma \):

\[
W = W(\epsilon_1, \epsilon_2, \epsilon_3, \gamma_1, \gamma_2, \gamma_3)
\]

where

\[
W = W(\epsilon_1, \epsilon_2, \epsilon_3, \gamma_1, \gamma_2, \gamma_3)
\]
\[ \frac{\tau I_{\phi}}{\mu} - \tau I_{\phi} = \tau \Pi_{I} \]

2. **Plane-Strain Stress Tensor**

We can then define hyperbolic stress measures as

A complete definition of the **Hyperbolic Stress Tensor** \( \tau I_{H} \) is given by Simo and Taylor [19].

\[
\begin{align*}
\tau I_{H} &= \left( \tau I_{H}(1-C) \right) \frac{\tau I_{H} + \tau I_{H} \tau I_{H}(1-C)}{1 + \tau I_{H}(1-C) \tau I_{H}(1-C)} \\
&\quad + \left( \tau I_{H} + \tau I_{H} \right) \frac{\tau I_{H} + \tau I_{H}}{1 + \tau I_{H}(1-C) \tau I_{H}(1-C)} \\
&\quad - \left( \tau I_{H}(1-C) \tau I_{H}(1-C) \right) \frac{\tau I_{H} + \tau I_{H}}{1 + \tau I_{H}(1-C) \tau I_{H}(1-C)} \\
&= \frac{\tau I_{H} \tau I_{H}}{\tau I_{H}} \Pi_{H} \tau I_{H}
\end{align*}
\]

We also define the **Second-Order Tensor** \( \tau I_{H} \) as

\[
\tau I_{H} = \tau I_{H} \tau I_{H}
\]

since from the orthogonality properties of the tensors,

\[
\tau I_{H} = \tau I_{H} \tau I_{H} = \tau I_{H}
\]

It can also be concluded that,

\[
\tau I_{H} + \tau I_{H} + \tau I_{H} = \tau I_{H}
\]

and it also follows

\[
\tau I_{H} = \tau I_{H}
\]

where was defined by equation \( \tau I_{H} \) was defined by equation \( \tau I_{H} \).

We obtain

\[
\text{from} \quad \left( \tau I_{H} \right) \left( \tau I_{H} - \frac{1}{2} I \right) - \tau I_{H} = \tau I_{H} - \left( \tau I_{H} \right) \left( \tau I_{H} - \frac{1}{2} I \right) = \left( \tau I_{H} \right) \left( \tau I_{H} - \frac{1}{2} I \right)
\]

In order to obtain these quantities we introduce a second order tensor

\[
\tau I_{H} = \tau I_{H}
\]

In order to obtain the second order tensors \( \tau I_{H} \) and other stress measures it is necessary to calculate the strain energy function \( \tau I_{H} \).

Moreover, the material tangent strain-stress tensor \( \tau I_{H} \) is given by the Hooke's law:

\[
\tau I_{H} = \frac{\tau I_{H}}{\mu}
\]
In the context of large deformation and application to continuum models, in the early works of Hill and Brown \cite{Hill1950} and Kronsber \cite{Kronsber1995} on inhomogeneities of crystal, the developments described here for the multiphase decomposition can be used as a kinematical basis for the multiphase decomposition of the deformation gradient.

\section*{3.2 Multiphase Decomposition}

\[(\nabla \mathbf{W}) \mathbf{v} \cdot \mathbf{n} \varepsilon + \nabla \mathbf{H} \cdot \mathbf{h} = \mathbf{T} \mathbf{M} \mathbf{I}_{\text{ref}} \]

\[(\nabla \mathbf{W}) \mathbf{v} \cdot \mathbf{n} \varepsilon + \nabla \mathbf{H} \cdot \mathbf{h} = \mathbf{T} \mathbf{M} \mathbf{I}_{\text{ref}} \]

where

\[(\nabla \mathbf{W}) \mathbf{v} \cdot \mathbf{n} \varepsilon + \nabla \mathbf{H} \cdot \mathbf{h} = \mathbf{T} \mathbf{M} \mathbf{I}_{\text{ref}} \]

The tangent stiffness operator is defined as

\[\mathbf{T} \mathbf{M} \mathbf{I}_{\text{ref}} = \mathbf{T} \mathbf{M} \mathbf{I}_{\text{ref}} \]

and

\[(\nabla \mathbf{W}) \mathbf{v} \cdot \mathbf{n} \varepsilon + \nabla \mathbf{H} \cdot \mathbf{h} = \mathbf{T} \mathbf{M} \mathbf{I}_{\text{ref}} \]

where

\[\mathbf{r} \mathbf{I} \mathbf{S} \mathbf{f} \mathbf{r} \mathbf{I} \mathbf{J} = \mathbf{r} \mathbf{I} \mathbf{J} \]

\[\mathbf{r} \mathbf{I} \mathbf{J} \mathbf{r} \mathbf{I} \mathbf{S} = \mathbf{r} \mathbf{I} \mathbf{J} \]

\[\mathbf{r} \mathbf{I} \mathbf{S} \mathbf{r} \mathbf{I} \mathbf{J} = \mathbf{r} \mathbf{I} \mathbf{J} \]
elastoplastic computations, the work by Lee and Liu [26], Fox [14] and Lee [25] generated an early interest in multiplicative decomposition.

The appropriateness of multiplicative decomposition technique for soils may be justified from the particulate nature of the material. From the micromechanical point of view, plastic deformation in soils arises from slipping, crushing, yielding and plastic bending\(^4\) of granules or platelets comprising the assembly\(^5\). It can certainly be argued that deformations in soils are predominantly plastic, however, reversible deformations could develop from the elasticity of individual soil grains, and could be relatively large, when particles are locked in high density specimens.

![Diagram of deformation gradient and multiplicative decomposition](image)

**Figure 2:** Multiplicative decomposition of deformation gradient: schematics.

The reasoning behind multiplicative decomposition is a rather simple one. If an infinitesimal neighborhood of a body \(x_i, x_i + dx_i\) in an inelastically deformed body is cut-out and unloaded to an unstressed configuration, it would be mapped into \(\hat{x}_i, \hat{x}_i + d\hat{x}_i\). The transformation would be comprised of a rigid body displacement\(^6\) and purely elastic unloading. The elastic unloading is fictitious, since in materials with a strong Baushinger’s effect unloading will lead to loading in another stress direction, and if there are residual stresses, the body must be cut-out in small pieces, and then every piece relieved of stresses. The unstressed

\(^4\)For plate like clay particles.

\(^5\)See also Borja and Alarcón [5] and Lambe and Whitman [24].

\(^6\)Translation and rotation.
\begin{equation}
0 \leq \frac{\nu y}{Y} \sqrt{\frac{\nu}{Y}} \sum f_1 f_2 = \mathcal{D}
\end{equation}

\[ (\nu y)_d \mathcal{M}^0 d + \left( \frac{f_1 f_2}{\nu y} \right)_d \mathcal{M}^0 d = (\nu y f_1 f_2)_d \mathcal{M}^0 d \]

We propose the free energy density \( \mathcal{M} \), which is defined in the intermediate configuration, \( \mathcal{I} \).

3.3 Constitutive Relations

3.3.1 Constitutive Relations in deformed body

The deformation gradient \( \mathcal{F} \) expresses the deformation of a body, not necessarily a continuous deformation. The material deformation gradient \( \mathcal{M} \) represents a suitable hyperelastic model in terms of the elastic rheol. The deformation part of the deformation gradient \( \mathcal{F} \) represents micro-mechanically the irreversible plastic deformation of a body, whereas the elastic part \( \mathcal{E} \) represents micro-mechanically the reversible elastic deformation.

\begin{equation}
\mathcal{F}_d \mathcal{I} \mathcal{F}_d = \mathcal{F}_d \mathcal{I} \mathcal{F}_d = \mathcal{F}_d \mathcal{I} \mathcal{F}_d
\end{equation}

where \( \mathcal{I} \) is not to be understood as a deformation gradient, since it may represent the intermediate configuration, etc.

\begin{equation}
\mathcal{F} \mathcal{I} \mathcal{F}_1 \mathcal{F}_1 (\frac{\nu y}{Y} \mathcal{M}^0 d) = \mathcal{F} \mathcal{I} \mathcal{F}_1 \mathcal{F}_1 = \mathcal{F} \mathcal{I} \mathcal{F}_1 \mathcal{F}_1
\end{equation}

We assume a linear relationship between \( \mathcal{F} \) and \( \mathcal{I} \).
\[ \left( \frac{\partial I_{t+u}}{\partial t} \right) \delta x = \frac{n}{\delta I_{t+u}} \]

and equation (73) we obtain

\[ \left( \frac{\partial I_{t}}{\partial t} \right) \frac{n}{\delta I_{t}} = \frac{n}{\delta I_{t}} = \frac{n}{\delta I_{t}} \]

By using the multiplicative decomposition

\[ \frac{n}{\delta I_{t}} \left( \frac{\partial I_{t+u}}{\partial t} \right) \delta x = \frac{n}{\delta I_{t+u}} \]

Therefore (67) and (69). The flow rule (67) can be interpreted to give The incremental deformation and plastic flow are governed by the system of evolution 

3.4 Implicit Integration Algorithm

where \( \gamma_n \) is the plastic part of the deformation gradient.

The constitutive relations can now be written as Hook’s elastic law is adopted. The constitutive relations, since we shall be dealing with small elastic deformations, are convenient, since they are derived from the constitutive relations and are consistent with the above analysis. The effective plastic strain can be expressed as

\[ 0 = \left( S \right)_{\text{eff}} = \left( \frac{\partial I_{t}}{\partial t} \right) \frac{n}{\delta I_{t}} = \left( \frac{\partial I_{t}}{\partial t} \right) \frac{n}{\delta I_{t}} = \left( \frac{\partial I_{t}}{\partial t} \right) \frac{n}{\delta I_{t}} \]

When yield function is isotropic in function of \( \phi \) and \( \psi \), we can conclude that

\[ 0 \leq \left( \frac{\partial I_{t}}{\partial t} \right) \frac{n}{\delta I_{t}} \]

We define the elastic domain is

\[ \left( \frac{\partial I_{t}}{\partial I_{t}} \right) \frac{n}{\delta I_{t}} \]

where the yielding stress and the plastic velocity defined on \( \frac{\partial I_{t}}{\partial I_{t}} \) now
\[
\frac{f_{\theta} \sigma_{\theta} + \phi}{1 + \sigma_{\theta}} + \phi = f_{\theta} \sigma_{\theta} + \phi
\]

collapses to

**Remark 3.2.** In the limit, when the deformations are sufficiently small, the solution (76)

Family of solutions for which are restricted to isotropic solids.

(76) is valid for a general anisotropic solid. Thus, the approximation of the general non-homogeneous tensor by equation of the Taylor's series expansion in equation (76) is a proper approximation for the Taylor's series expansion above.

First order series expansion includes constant and linear (up to linear) terms. Second order

(76)

\[
\left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial y} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial x} - \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial z} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial y} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial x} - \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial z}
\]

obtained by using the second order expansion in equation (76) and after some tensor algebra we

(79)

\[
\cdots + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial y} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial x} - \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial z} = \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial y} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial x} - \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial z}
\]

By recognizing that the expression of a tensor can be expanded in Taylor series

(79)

\[
\left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial y} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial x} - \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial z} = \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial y} + \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial x} - \left( \frac{f_{\theta}}{1 + \sigma_{\theta}} \right) f_{\theta} \sigma_{\theta} \frac{\partial}{\partial z}
\]

The elastic deformation is then

(79)

\[
1 - \left( \frac{\partial}{\partial y} \right) \left( \frac{w}{u} \right) \frac{\partial}{\partial x} = \frac{\partial}{\partial y} \frac{\partial}{\partial x}
\]

where we used that
\[(81)\]
\[
(n^T \frac{\partial g}{\partial t}) \Phi = \Phi
\]

where

\[(83)\]
\[
0 = \Phi_{t+u} \frac{\partial \nu}{\partial t} \quad ; \quad 0 \gg \Phi_{t+u} \quad ; \quad 0 > \nu \frac{\partial \nu}{\partial t}
\]

and the Karush-Kuhn-Tucker (KKT) conditions

\[(82)\]
\[
\frac{n_{t+u} \nu \theta}{M} + \frac{\nu u}{M} = \frac{n_y}{M}
\]

\[(88)\]
\[
\frac{n_{t+u} \nu \theta}{M} \frac{\partial \nu}{\partial t} + \nu u = \frac{n_y}{M}
\]

The incremental problem is defined by equations (79), (80) and the constitutive relations

\[(69)\]
\[
\left( \frac{n_{I} I_{t+u} n_{t+u} p_{t+u} + \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} \right) \frac{\partial \nu}{\partial t} - \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} = \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t}
\]

The hardening rule (69) can be integrated to give

\[(70)\]
\[
\left( \frac{n_{I} I_{t+u} n_{t+u} p_{t+u} + \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} \right) \frac{\partial \nu}{\partial t} - \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} = \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t}
\]

The elastic deformation tensor can be written as

\[(71)\]
\[
\frac{n_{I} I_{t+u} n_{t+u} p_{t+u} + \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} \right) \frac{\partial \nu}{\partial t} - \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} = \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t}
\]

By neglecting the higher order term with \( \frac{\partial \nu}{\partial t} \) in equation (70), the solution for the right

\[(72)\]
\[
\frac{n_{I} I_{t+u} n_{t+u} p_{t+u} + \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} \right) \frac{\partial \nu}{\partial t} - \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} = \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t}
\]

In working out the small deformation component (72) it was used that

\[(77)\]
\[
\frac{n_{I} I_{t+u} n_{t+u} p_{t+u} + \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} \right) \frac{\partial \nu}{\partial t} - \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t} = \frac{\mu_1}{\alpha} \frac{\partial I_{t+u}}{\partial t}
\]

which represents a small deformation elastic predictor-plastic corrector equation in strain
\[
\left( f^Z_1 + u \right) \nabla - f^\text{pred}_{\Delta \omega_2 1+u} = f^Z_2
\]

The new residual \( f^Z_2 \) from the old \( f^Z_1 \).

The new residual expression can be applied to the tensor of residuals in order to obtain the iterative change, \( f^Z_{1+u} \).

The first order Taylor series expansion is maintained fixed during the iteration process. The first order Taylor series tensor \( f^Z_{1+u} \) is treated as the right elastic deformation tensor. The initial right elastic deformation tensor and the back-calculated initial right elastic deformation tensor represent the difference between the current right elastic deformation tensor.

The tensor \( f^Z_2 \) represents the difference between the current right elastic deformation tensor.

\[
\left( f^Z_{1+u} \nabla - \frac{f^Z_{1+u}}{\partial \Delta \omega_2 1+u} \right) - \frac{f^Z_2}{2} = f^Z_2
\]

We introduce a tensor of deformation residuals

\[
\left( \frac{s^f_d d u}{s^d d u} \right) \left( f^Z_{1+u} \nabla - \frac{f^Z_{1+u}}{\partial \Delta \omega_2 1+u} \right) = \left( f^Z_2 \nabla - \frac{f^Z_2}{\partial \Delta \omega_2 1+u} \right)
\]

We introduce the above assumption of first order expansion in \( f^Z_1 \), the initial right elastic deformation tensor.

\[
\left( f^Z_1 \nabla - \frac{f^Z_{1+u}}{\partial \Delta \omega_2 1+u} \right) \nabla = f^Z_2
\]

\( f^Z_2 \) have introduced tensor plastic correction equations.

We used as a starting point for a Newton iterative algorithm. In the previous equation, we

\[
\left( f^Z_{1+u} \nabla - \frac{f^Z_{1+u}}{\partial \Delta \omega_2 1+u} \right)
\]

The classical plastic correction equation

\[
\left( f^Z_{1+u} \nabla - \frac{f^Z_{1+u}}{\partial \Delta \omega_2 1+u} \right) = \frac{f^Z_{2+u}}{\partial \Delta \omega_2 1+u}
\]

\[
\left( f^Z_{1+u} \nabla - \frac{f^Z_{1+u}}{\partial \Delta \omega_2 1+u} \right) = \frac{f^Z_{2+u}}{\partial \Delta \omega_2 1+u}
\]

We used the appropriate pull-back or push-forward to obtain \( f^Z_{1+u} \) and \( f^Z_{2+u} \) from \( f^Z_{1+u} \) and \( f^Z_{2+u} \). The updated quantities \( f^Z_{1+u} \) and \( f^Z_{2+u} \) can be obtained from the second plane-strain hydrostatic stress tensor and the right elastic deformation tensor. The updated stress tensor \( f^Z_{1+u} \) and the right elastic deformation tensor \( f^Z_{1+u} \) can be obtained from the second plane-strain hydrostatic stress tensor.

\[ f^Z_{1+u} = \frac{f^Z_{1}}{2} \frac{f^Z_{2}}{2} \]

Remark 3.3: The updated stress tensor \( f^Z_{2+u} \) and the right elastic deformation tensor \( f^Z_{1+u} \) can be obtained from the second plane-strain hydrostatic stress tensor.
\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) + \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p - \left( \frac{\eta H}{\Phi} \right) p = \left( \frac{\eta H}{\Phi} \right) p
\]

A first order Taylor series expansion of a yield function together with (26) provides

\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p - \left( \frac{\eta H}{\Phi} \right) p = \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \varepsilon\Phi p
\]

By using that

\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p - \left( \frac{\eta H}{\Phi} \right) p = \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \varepsilon\Phi p
\]

we can solve for \( \eta H \) introducing the notation

\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p - \left( \frac{\eta H}{\Phi} \right) p = \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \varepsilon\Phi p
\]

so that after setting \( 0 = \frac{\varepsilon\Phi}{\Phi} \) and some tensor algebra we obtain

\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \varepsilon\Phi p
\]

so that after setting \( 0 = \frac{\varepsilon\Phi}{\Phi} \) and some tensor algebra we obtain

\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \varepsilon\Phi p
\]

we can write

\[
(66) \quad \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \frac{\varepsilon\Phi}{\Phi} \left( \frac{\varepsilon\Phi \varepsilon H}{\Phi^2} \right) \left( \frac{\eta H}{\Phi} \right) p = \varepsilon\Phi p
\]

Furthermore, since
The procedure described below summarizes the implementation of the return algorithm.

\[ (101) \]

\[ \frac{b_{d_1} \tilde{z}}{\phi_{\Phi \theta} - \frac{\partial \Phi_{\theta}}{\partial \tilde{z}}} = \frac{b_{d_1} \tilde{z}}{u_{u}^{H}} \]

\[ \frac{b_{d_{lu}} \tilde{z}}{u_{u}^{H}} = \frac{b_{d_{lu}} \tilde{z}}{u_{u}^{H}} \]

\[ \frac{b_{d_{lu}}}{1 + \frac{\tilde{z}}{u_{u}^{H}}} = \frac{b_{d_{lu}}}{1 + \frac{\tilde{z}}{u_{u}^{H}}} \]

By noting that the residual is defined in stream space, the incremental consistency

since in the limit, as deformations become small

\[ (103) \]

\[ \frac{u_{u}^{H} + u_{d_{lu}}}{1 - \left( \frac{b_{d_{lu}}}{u_{u}^{H}} \right) \left( \frac{b_{d_{lu}}}{u_{u}^{H}} \right) \left( \frac{b_{d_{lu}}}{u_{u}^{H}} \right) - \Phi_{\eta \Phi} \] = \( (\eta_{\nabla})p \]

Remark 3.4: In the limit, for small deformations, the incremental consistency parameter

\[ (102) \]

\[ \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} = \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \]

\[ u_{u}^{H} + u_{d_{lu}} \]

\[ 1 - \left( \frac{b_{d_{lu}}}{u_{u}^{H}} \right) \left( \frac{b_{d_{lu}}}{u_{u}^{H}} \right) \left( \frac{b_{d_{lu}}}{u_{u}^{H}} \right) - \Phi_{\eta \Phi} \]

where setting \( u_{u}^{H} = \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \) we can solve for the incremental consistency parameter

\[ (101) \]

\[ \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} = \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \]

\[ \left( \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \right) \left( \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \right) \left( \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \right) \left( \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \right) = \frac{\partial \Phi_{\theta}}{\partial \tilde{z}} \]

and with the solution for \( b_{d_{lu}} \)

\[ (86) \]

\[ (86) \]

\[ (86) \]

\[ (86) \]

\[ (86) \]

\[ (86) \]
24

\[ \frac{f_{2}}{\Phi} = \frac{f_{2}}{\Phi} + u \]

is satisfied set \( T \). For convergence criterion \( TOLN \geq \| \frac{f_{2}}{\Phi} \| \) and \( TOLN \geq (\frac{f_{2}}{\Phi} \Phi \)

Step 2: Check for convergence, known variables

\[ \begin{align*}
(\frac{f_{\eta}Z_{t+u} + v}{\Phi} - \frac{f_{\eta}2_{t+u} + v}{\Phi}) - (\frac{f_{\eta}Z_{t+u} + v}{\Phi} - \frac{f_{\eta}2_{t+u} + v}{\Phi}) &= \frac{f_{\eta}}{\Phi} \\
(\frac{v}{\Phi} + \frac{f_{\eta}2_{t+u} + v}{\Phi}) &= \frac{f_{\eta}}{\Phi} \\
(\frac{\eta}{\Phi} + \frac{f_{\eta}2_{t+u} + v}{\Phi}) &= \frac{f_{\eta}}{\Phi} \\
\end{align*} \]

Evaluate the yield function and the residual

Step 3: Iteration. Known variables

If the yield criterion has been violated \((0 < \Phi \Phi_{t+u} \geq 0)\) then proceed.

Current Increment

and exit constitutive iteration procedure, there is no plastic flow in the

\[ \begin{align*}
\frac{\eta}{\Phi}L_{t+u} &= \frac{f_{\eta}}{\Phi} L_{t+u} \\
\frac{\Phi}{\Phi}Y_{t+u} &= \frac{f_{\eta}}{\Phi} Y_{t+u} \\
\frac{\Phi}{\Phi}2_{t+u} &= \frac{f_{\eta}}{\Phi} 2_{t+u} \\
\end{align*} \]

We then evaluate the yield function \( \Phi \Phi_{t+u} \geq 0 \) and set

\[ \begin{align*}
\frac{f_{\eta}}{\Phi}L_{t+u} + \frac{f_{\eta}}{\Phi}2_{t+u} &= \frac{f_{\eta}}{\Phi}L_{t+u} \\
\frac{f_{\eta}}{\Phi}2_{t+u} &= \frac{f_{\eta}}{\Phi}2_{t+u} \\
\end{align*} \]

Stress Tensor

Then we compute the initial second Piola-Kirchhoff stress and the initial strain

\[ \begin{align*}
\left( \frac{f_{\eta}}{\Phi}f_{t+u} \right)_{J} \left( \frac{f_{\eta}}{\Phi}f_{t+u} \right)_{J} \left( \frac{f_{\eta}}{\Phi}f_{t+u} \right)_{J} + \left( \frac{f_{\eta}}{\Phi}f_{t+u} \right)_{J} = \frac{f_{\eta}}{\Phi}2_{t+u} \\
\end{align*} \]

and the right deformation tensor

\[ \begin{align*}
\frac{f_{\eta}}{\Phi}n + \frac{f_{\eta}}{\Phi} = \frac{f_{\eta}}{\Phi}f_{t+u} \\
\end{align*} \]

\[ \begin{align*}
\frac{f_{\eta}}{\Phi} \nabla_{t+u} \text{ for a given displacement increment specific quadrature point in a finite element we compute the relative deformation gradient} \\
\text{Given the initial elastic deformation tensor and a set of hardening variables at a} \\
\text{Return Algorithm} \\
\]
Briefly we are omitting superscript $\mathbb{P}$.

From step 3, to step 9, all of the variables are in intermediate $u + I$ configuration. For the sake of convenience we will express the terms with a superscript $\mathbb{P}$.

\[
(121) \quad \left( \frac{\gamma}{\gamma} H \left( (1 + \gamma) \nabla \right) \right) \left( \frac{\gamma}{\gamma} \nabla \right) \left( (1 + \gamma) \nabla \right) p + \frac{\gamma}{\gamma} \nabla \left( \frac{\gamma}{\gamma} \nabla \right) \left( (1 + \gamma) \nabla \right) + \frac{\gamma}{\gamma} Z \left( 1 + u \right) \left( (1 + \gamma) \nabla \right) p - \frac{\gamma}{\gamma} \nabla \left( (1 + \gamma) \nabla \right)
\]

\[
\left( \frac{\gamma}{\gamma} \nabla \right) \left( \frac{\gamma}{\gamma} \nabla \right) = (1 + \gamma) \nabla
\]

\[
\text{and the Mises stress}
\]

\[
\text{Step 6. Calculate the increments for the right deformation tensor, the hardening variable}
\]

\[
\text{(111) } (1 + \gamma) \nabla \left( (1 + \gamma) \nabla \right) p + (\gamma) \nabla = (1 + \gamma) \nabla
\]

\[
\text{Step 6. Update the consistency parameter}
\]

\[
\frac{\gamma}{\gamma} L \left( (1 + \gamma) \nabla \right) + \frac{\gamma}{\gamma} L \left( (1 + \gamma) \nabla \right) \left( (1 + \gamma) \nabla \right) p + \frac{\gamma}{\gamma} \nabla \left( (1 + \gamma) \nabla \right) \left( (1 + \gamma) \nabla \right) = \frac{\gamma}{\gamma} \nabla
\]

\[
\left( (1 + \gamma) \nabla \right) \left( \frac{\gamma}{\gamma} \nabla \right) = (1 + \gamma) \nabla
\]

\[
\frac{\gamma}{\gamma} H \left( (1 + \gamma) \nabla \right) + \frac{\gamma}{\gamma} H \left( (1 + \gamma) \nabla \right) \left( (1 + \gamma) \nabla \right) p - \frac{\gamma}{\gamma} \nabla \left( (1 + \gamma) \nabla \right)
\]

\[
\left( \frac{\gamma}{\gamma} \nabla \right) \left( \frac{\gamma}{\gamma} \nabla \right) = (1 + \gamma) \nabla
\]

\[
\text{Step a. Compute the incremental consistency parameter}
\]

\[
\text{(011) } (1 + \gamma) \nabla \left( (1 + \gamma) \nabla \right) p + (\gamma) \nabla \left( (1 + \gamma) \nabla \right) \left( (1 + \gamma) \nabla \right) = (1 + \gamma) \nabla
\]

\[
\frac{\gamma}{\gamma} H \left( (1 + \gamma) \nabla \right) + \frac{\gamma}{\gamma} H \left( (1 + \gamma) \nabla \right) \left( (1 + \gamma) \nabla \right) p - \frac{\gamma}{\gamma} \nabla \left( (1 + \gamma) \nabla \right)
\]

\[
\left( \frac{\gamma}{\gamma} \nabla \right) \left( \frac{\gamma}{\gamma} \nabla \right) = (1 + \gamma) \nabla
\]

\[
\text{Step 3. If convergence is not achieved compute the elastic stiffness tensor}
\]

\[
\text{Elastic constitutive integration procedure}
\]

\[
(\gamma) \nabla_{I} + u = (\gamma) \nabla_{I} + u
\]

\[
(\gamma) L_{I} + u = (\gamma) L_{I} + u
\]

\[
(\gamma) Y_{I} + u = (\gamma) Y_{I} + u
\]

\[
(\gamma) Y_{I} + u = (\gamma) Y_{I} + u
\]
\[ f_{11} \nabla_{t} \mathbf{u} - f_{12} \mathbf{a}_{2t} = f_{12} \mathbf{a}_{2t+u} \]

Starting from the elastic predictor-plastic corrector equation

### 3.5

**Algorithms: Tangent Stiffness Tensor**

and return to step 2.

\[ \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} = \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} \]
\[ \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} = \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \]
\[ \frac{\text{v}_{y} y}{(1+\gamma)^{I}} = \frac{\text{v}_{y} y}{(1+\gamma)^{I}} \]
\[ \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} = \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} \]
\[ \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} = \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} \]

**Step 9.** Set \( \gamma = 1 \) and

\[ \left( \frac{f_{11}}{(1+\gamma)^{I}} \right) \left( \frac{f_{11}}{(1+\gamma)^{I}} \nabla_{t} \mathbf{u} - \frac{f_{12}}{(1+\gamma)^{I}} \mathbf{a}_{2t} \right) \left( \frac{f_{11}}{(1+\gamma)^{I}} \nabla_{t} \mathbf{u} - \frac{f_{12}}{(1+\gamma)^{I}} \mathbf{a}_{2t} \right) = \frac{f_{11}}{(1+\gamma)^{I}} Y \]

\[ \left( \frac{f_{11}}{(1+\gamma)^{I}} Y \right) \left( \frac{f_{11}}{(1+\gamma)^{I}} \nabla_{t} \mathbf{u} - \frac{f_{12}}{(1+\gamma)^{I}} \mathbf{a}_{2t} \right) = \frac{f_{11}}{(1+\gamma)^{I}} \Phi \]

**Step 8.** Evaluate the yield function and the residual

\[ \left( \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} \right) p + \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} = \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} \]
\[ \left( \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \right) p + \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} = \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \]
\[ \left( \frac{\text{v}_{y} y}{(1+\gamma)^{I}} \right) p + \frac{\text{v}_{y} y}{(1+\gamma)^{I}} = \frac{\text{v}_{y} y}{(1+\gamma)^{I}} \]
\[ \left( \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} \right) p + \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} = \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} \]

**Handed stress tensor**

\[ \text{Handed stress tensor} \]

**Step 7.** Update the residual deformation tensor variable \( \text{bd}_{I}^{y} 2 \) and return to step 2.

\[ \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} 2 p + \frac{\text{bd}_{I}^{y} 2}{(1+\gamma)^{y} 2} 2 p + \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} = \frac{\text{um}_{I}^{L}}{(1+\gamma)^{I}} \]

\[ \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} H \left( \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \right) p = \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \]

\[ \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} = \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \]

\[ \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \frac{\text{v}_{y} Y}{(1+\gamma)^{I}} \]
\[
\mathcal{L}_2
\]

\[
\mathcal{L}_2 p_{\mu \nu}^a \partial_a \mathcal{L} = \frac{\partial}{\partial \mathcal{L}}
\]

Then

\[
\mathcal{L}_2
\]

\[
\frac{\partial}{\partial \mathcal{L}} \left( \frac{\varepsilon \frac{\partial}{\partial \mathcal{L}}}{\Phi \mathcal{L}} + \frac{\varepsilon \frac{\partial}{\partial \mathcal{L}}}{\Phi \mathcal{L}} \right) \mathcal{L} = \frac{\partial}{\partial \mathcal{L}} \left( \frac{\varepsilon \frac{\partial}{\partial \mathcal{L}}}{\Phi \mathcal{L}} \right) \mathcal{L}
\]

and by using we can write

\[
\mathcal{L}_2 p_{\mu \nu}^a \partial_a \mathcal{L} = \frac{\partial}{\partial \mathcal{L}}
\]

Since

\[
\mathcal{L}_2
\]

\[
\mathcal{L}_2 p_{\mu \nu}^a \partial_a \mathcal{L} = \frac{\partial}{\partial \mathcal{L}}
\]

We are now in the position to solve for the incremental consistency parameter

\[
\mathcal{L}_2
\]

\[
\mathcal{L}_2 p_{\mu \nu}^a \partial_a \mathcal{L} = \frac{\partial}{\partial \mathcal{L}}
\]

By using the solution for from (001) we can write

\[
0 = \frac{\partial}{\partial \mathcal{L}} \left( \frac{\varepsilon \frac{\partial}{\partial \mathcal{L}}}{\Phi \mathcal{L}} \right) \mathcal{L}
\]

with defined in (001).

Next we use the first order Taylor series expansion of the yield function

\[
\mathcal{L}_2
\]

where was defined in (f\mu L)

\[
\mathcal{L}_2
\]

in which we apply a first order Taylor series expansion to obtain (after some tensor algebra)

\[
\mathcal{L}_2
\]
3.6 Material Model

with its small strain counterpart (Jaime and Simeon [19]) is given by (131) completely exactly:

\[
\frac{\partial \mathbf{Y}}{\partial \mathbf{E}} + \mathbf{u} \frac{\partial \mathbf{Y}}{\partial \mathbf{u}} = \mathbf{L} \mathbf{L}^{-1}
\]

\[
\frac{\partial \mathbf{Y}}{\partial \mathbf{E}} + \mathbf{u} \frac{\partial \mathbf{Y}}{\partial \mathbf{u}} \nabla \mathbf{u} = \mathbf{L} \mathbf{L}^{-1}
\]

\[
\mathbf{L} = \frac{\partial \mathbf{Y}}{\partial \mathbf{E}} + \mathbf{u} \frac{\partial \mathbf{Y}}{\partial \mathbf{u}}
\]

\[
\mathbf{L} \mathbf{L}^{-1} = \mathbf{L}^{-1} \mathbf{L}
\]

Since

\[
\mathbf{L} \mathbf{L}^{-1} = \mathbf{L}^{-1} \mathbf{L}
\]

Tangent stiffness tensor becomes

**Remark 3.6.1** In the limit for small deformations and isotropic response, the algorithmic

\[
\mathbf{L} = \frac{\partial \mathbf{Y}}{\partial \mathbf{E}} + \mathbf{u} \frac{\partial \mathbf{Y}}{\partial \mathbf{u}}
\]

\[
\mathbf{L} = \mathbf{L}^{-1} \mathbf{L}
\]

Pull-back to the reference configuration is given by

\[
\mathbf{L} = \mathbf{L}^{-1} \mathbf{L}
\]

defined as

\[
\mathbf{L} = \mathbf{L}^{-1} \mathbf{L}
\]

The algorithmic tangent stiffness tensor \( \mathbf{L} \) in intermediate configuration is then

\[
\left( \frac{\partial \mathbf{Y}}{\partial \mathbf{E}} + \mathbf{u} \frac{\partial \mathbf{Y}}{\partial \mathbf{u}} \nabla \mathbf{u} \right) \left( \frac{1}{\mathbf{L}^{-1} \mathbf{L}} \mathbf{L}^{-1} \mathbf{L} \right) \left( \mathbf{L}^{-1} \mathbf{L} \right) = \mathbf{L}^{-1} \mathbf{L}
\]

where
we model the experiments with a 3D model. Six quadrangle 20-node brick elements were
used for a full three-dimensional implementation. Although the size of stress is enforced,
the three-dimensional finite element mesh used to model the MGM is deleted in

Figure 4. Instead of developing two-dimensional finite element formulation, we have

Detailed description of the experimental setup is given by Srinivasan et al. [20].

The elastic response appears to be very stiff (from unloading-reloading loops).

The tests. The strain contains significant noise and the presented data are in

Figure 3. Micro Gravity Mechanics, load-displacement and volume displacement curves for

In this section we present numerical modeling of low confinement, microgravity, large-de-

4 Numerical Simulations of Micro Gravity Mechanics

Figure 3 shows load-displacement and volume-displacement data for three low confinement

formation than the test performed during space shuttle STS-79 mission in September 1996.

and potential surfaces. A detailed description of the model is given by Jereme et al. [18].

deformation mode definition is based on the use of the yield stress. For describing yield

during micro gravity mechanics tests aboard space shuttle (Srinivasan et al. [20]), the large

modulated and the yield surface was shaped in such a way to mimic recent findings obtained

and is subsequently denoted the D-Model. The D-Model is a single surface model with un-
Special attention was given to the specimen ends, where the layer membrane

spring stiffness. Special attention was given to the specimen ends, where the layer membrane

interaction of the stiffness terms for the quadratic brick element then supplied equivalent
numerical specimen where used to form a non-linear finite element analysis model. The
spring method. The output from the one element extension tests on the hydraulic test
spring stiffness ratio would be 1.3.3 mm² with 8 0.3 mm², 0.3 mm² in parallel to the
boundary nodes. Instead of using thin, highly distorted brick elements (membrane in the
matrix), the membrane influence was modeled by adding equivalent stiffness (springs) to the
load nodes by means of equivalent forces, obtained through the partial inversion of a stiffness
top node over the membrane boundary condition. Applied displacements to the
the movable boundary at the top. The top movable boundary applied displacement to the
After the first stage, the displacement boundary conditions were changed by adding

unit hyperelastic.

membrane pressure had a minor effect at this stage. During this stage, the response was
was removed, since the membrane does not have significant stiffness in compression; and
symmetry displacement boundary conditions were in place. Influence of the membrane
the first stage involved isotropic compression in the design pressure. For the first stage only
chosen to model one-eighth of the specimen. The analysis was performed in two stages.

Figure 4: Finite element mesh for the MGA specimen.
was wrapped around the end platen and created a ring in the horizontal plane (parallel to end platen) which was stiffer than the unstretched membrane surrounding the specimen. The last row of nodes was thus supported by stiffer equivalent membrane elements. The material parameters for the B Material Model for all three confining pressures\textsuperscript{12} were kept the same except for the Young’s modulus. This consistency in material parameters is important, since all three specimens contained the same Ottawa F-75 sand at 85% relative density.

![Graph](image1)

**Figure 5:** Mechanics of granular materials responses, initial confinement ($p_0 = 0.05kPa$) test (a) load–deformation and (b) volume–deformation experiments and numerical results.

![Graph](image2)

**Figure 6:** Mechanics of granular materials responses, initial confinement ($p_0 = 0.52kPa$) test (a) load–deformation and (b) volume–deformation experiments and numerical results.

\textsuperscript{12} E = 300.0; 360.0; 700.0 kN/m\textsuperscript{2}; \nu = 0.2 ; p_c = 1000.0 kN/m\textsuperscript{2}; p_t = 0.0 kN/m\textsuperscript{2}; n = 0.2 ; a = 5.0 ; b = 0.707 ; \eta_{init} = 2.5 ; b_1 = 1.0 ; d_{hard} = 5000.0 ; e_{hard} = 0.5 ; \eta_{res} = 0.15 ; \eta_{peak} = 1.75 ; \eta_{start} = 0.25 ; l = 1.0 ; c_{conc} = 0.030 ; r = 1.00 ; c_{cap} = 0.30 ; p_{c,0} = 1000.0 kN/m\textsuperscript{2}; a_s = 100.0 ; b_s = 0.707)
Figure 7: Mechanics of granular materials responses, initial confinement ($p_0 = 1.30\text{kPa}$) test (a) load–deformation and (b) volume–deformation experiments and numerical results.

Figures 5(a), 6(a) and 7(a) show comparison of numerical modeling with the test data for load–displacement. Following observations are made. The initial (elastic) stiffness is higher in the actual experiments. The peak strength is modeled quite accurately, while the post–peak behavior is slightly stiffer in the numerical experiment. The residual stiffness is softer in the numerical model than observed in the MGM tests. This can be explained by the stiffer specimen ends in a physical test. In other words, the latex membrane wrapped around the end platens (the end platens are 30% wider than the specimen) usually sticks to the end platen after some radial displacements and then acts as a full restraint. The friction between end platens and the sand specimen can also add to the whole specimen stiffness, however, the end platens were made of highly polished tungsten–carbide, which has a very low friction angle with quartz sand (3°), and we have thus decided to neglect the influence of end platen friction on the overall response. It is of interest to note that the maximum mobilized friction angle is in the range of 70° and the dilatancy angles observed in the early parts of the experiments are 30°, which is unusually high.

Figures 5(b), 6(b) and 7(b) shows comparison of volumetric–displacement data for experiments and numerical modeling. In modeling the lowest confinement ($p_0 = 0.05\text{kPa}$) level we correctly predict complete lack of volumetric compression. Numerical predictions for two other confinements ($p_0 = 0.52\text{kPa}$, $p_0 = 1.20\text{kPa}$) shows small amount of initial volume compression which was not observed in experiments. Figure 8 shows a typical specimen before and after the test. The latex ring formed by wrapping of membrane around end platens is visible on both specimen ends.
Figure 12 depicts the deformed shape of a specimen. Without the latex membrane, the

Figure 8: The specimen before and after the test.
5 Concluding Remarks

The deformed shape shown in Figure 12 demonstrates the effect of the difference in initial curvature of the membrane on the overall response. The above-mentioned results are in agreement with the finite element analysis performed in this study.

Figures 10 and 11: Comparison of experimental and numerical results for (a) load-displacement and (b) load-volume change, with and without membrane. The initial curvature of the membrane was varied, with a peak load of 0 kN.
stress and strains is lost (for anisotropic and cyclic response). A detailed constitutive formulation has been presented. Moreover, the return algorithm was outlined with implementation details. The developed formulation and implementation were used to simulate large deformation tests on sand performed under very low confinement. To this end, a consistent set
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Behavior of sand specimen at very low confinement pressures, it was shown that the latter membrane has substantial influence on the confinement losses. The model for the material behavior used to simulate these low confinement pressures for the D material model was used to accurately simulate these low confinement pressures.
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