Stochastic Elastic-Plastic Finite Element Method: Recent Advances and Developments

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Крагујевац, Србија
Outline

Introduction

Probabilistic Inelastic Modeling
  FPK Formulations
  Direct Solution for Probabilistic Stiffness and Stress

Summary
## Motivation

- Probabilistic fish counting
- Williams’ DEM simulations, differential displacement vortices
- SFEM round table
- Kavvas’ probabilistic hydrology
- Performance based design, probability of undesirable performance, \((10^{-4}, 10^{-5})\)
Motivation

Material Behavior Inherently Uncertain

- Spatial variability
- Point-wise uncertainty, testing error, transformation error

(Mayne et al. (2000))
Parametric Uncertainty: Material and Loads

Transformation of SPT $N$-value: 1-D Young’s modulus, $E$ (cf. Phoon and Kulhawy (1999B))

$$E = (101.125 \times 19.3) N^{0.63}$$
Parametric Uncertainty: Material Properties
It is not a new Problem

Le doute n’est pas un état bien agréable, mais l’assurance est un état ridicule.

François-Marie Arouet (Voltaire)
Types of Uncertainties

- Epistemic uncertainty - due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools are not well developed
  - trade-off with aleatory uncertainty

- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools

- Ergodic Assumption!?
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Uncertainty Propagation through Inelastic System

- Incremental el–pl constitutive equation

\[ \Delta \sigma_{ij} = E_{ijkl}^{EP} \Delta \epsilon_{kl} = E_{ijkl}^{el} - \frac{E_{ijmn}^{el} m_{mn} n_{pq} E_{pqkl}^{el}}{n_{rs} E_{rstu}^{el} m_{tu} - \xi^* h^*} \Delta \epsilon_{kl} \]

- Dynamic Finite Elements

\[ \ddot{M}u + \dot{C}u + K^{ep}u = F \]

- What if all (any) material and load parameters are uncertain
Probabilistic Elastic-Plastic Response

![Graph showing stress-strain relationship with various lines indicating mean, mode, and standard deviations.](image)

- **Mean** line represents the average stress-strain response.
- **Mode** line indicates the peak stress at a given strain.
- **Deterministic Solution** line shows the expected stress-strain response without variability.
- **Std. Deviations** lines illustrate the variability around the mean stress-strain response.

**FPK Formulations**

**Probabilistic Inelastic Modeling**

*Jeremić and Lacour*

**SEPFEM**
Previous Work

- Linear algebraic or differential equations:
  - Cumulant Expansion Method (Gardiner 2004)

- Nonlinear differential equations:
    → can be accurate, very costly
    → first and second order Taylor series expansion about mean - limited to problems with small C.O.V. and inherits "closure problem"
3D FPK Equation

\[
\frac{\partial P(\sigma_{ij}(x_t, t), t)}{\partial t} = \frac{\partial}{\partial \sigma_{mn}} \left\{ \left\langle \eta_{mn}(\sigma_{mn}(x_t, t), D_{mnrs}(x_t), \epsilon_{rs}(x_t, t)) \right\rangle \right. \\
+ \left. \int_0^t d\tau \text{Cov}_0 \left[ \frac{\partial \eta_{mn}(\sigma_{mn}(x_t, t), D_{mnrs}(x_t), \epsilon_{rs}(x_t, t))}{\partial \sigma_{ab}} \right. \left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t - \tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t - \tau)) \right] \right\} P(\sigma_{ij}(x_t, t), t) \right. \\
+ \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[ \left\{ \int_0^t d\tau \text{Cov}_0 \left[ \eta_{mn}(\sigma_{mn}(x_t, t), D_{mnrs}(x_t), \epsilon_{rs}(x_t, t)) \right. \left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t - \tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t - \tau)) \right] \right. \right. \left\} \right. P(\sigma_{ij}(x_t, t), t) \left. \right. \]
FPK Formulations

FPK Equation

- Advection-diffusion equation

\[
\frac{\partial P(\sigma, t)}{\partial t} = - \frac{\partial}{\partial \sigma} \left[ N(1) P(\sigma, t) - \frac{\partial}{\partial \sigma} \{ N(2) P(\sigma, t) \} \right]
\]

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- It is deterministic equation in probability density space
- It is linear PDE in probability density space \(\rightarrow\) simplifies the numerical solution process
Template Solution of FPK Equation

- FPK diffusion–advection equation is applicable to any material model → only the coefficients $N_{(1)}$ and $N_{(2)}$ are different for different material models

- Initial condition
  - Deterministic → Dirac delta function → $P(\sigma, 0) = \delta(\sigma)$
  - Random → Any given distribution

- Boundary condition: Reflecting BC → conserves probability mass $\zeta(\sigma, t)|_{\text{At Boundaries}} = 0$

- Solve using finite differences and/or finite elements

- However (!!) it is a stress solution and probabilistic stiffness is an approximation!
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SEPFEM
Direct Probabilistic Constitutive Modeling

- Zero elastic region elasto-plasticity with stochastic Armstrong-Frederick kinematic hardening
  \[ \Delta \sigma = H_a \Delta \varepsilon - c_r \sigma |\Delta \varepsilon|; \quad E_t = d\sigma/d\varepsilon = H_a \pm c_r \sigma \]

- Uncertain: init. stiff. \( H_a \), shear strength \( H_a/c_r \), strain \( \Delta \varepsilon \):
  \[ H_a = \sum h_i \Phi_i; \quad C_r = \sum c_i \Phi_i; \quad \Delta \varepsilon = \sum \Delta \varepsilon_i \Phi_i \]

- Resulting stress and stiffness are also uncertain
Direct Solution for Probabilistic Stiffness and Stress

Probabilistic Stiffness Solution

- **Analytic product for all the components,**

\[
E_{ijkl}^{EP} = \left[ E_{ijkl}^{el} - \frac{E_{ijmn}^{el} m_{mn} n_{pq} E_{pqkl}^{el}}{n_{rs} E_{rstu}^{el} m_{tu} - \xi^* h^*} \right]
\]

- **Stiffness: each Polynomial Chaos component is updated incrementally**

\[
E_{t_1}^{n+1} = \frac{1}{<\Phi_1\Phi_1>} \{ \sum_{i=1}^{P_h} h_i < \Phi_i\Phi_1 > \pm \sum_{j=1}^{P_c} \sum_{l=1}^{P_{\sigma}} c_j \sigma_j^{n+1} < \Phi_j\Phi_j\Phi_1 > \}
\]

\[
\vdots
\]

\[
E_{t_P}^{n+1} = \frac{1}{<\Phi_1\Phi_P>} \{ \sum_{i=1}^{P_h} h_i < \Phi_i\Phi_P > \pm \sum_{j=1}^{P_c} \sum_{l=1}^{P_{\sigma}} c_j \sigma_j^{n+1} < \Phi_j\Phi_j\Phi_P > \}
\]

- **Total stiffness is**

\[
E_t^{n+1} = \sum_{i=1}^{P_E} E_{t_i}^{n+1} \Phi_i
\]
Direct Solution for Probabilistic Stiffness and Stress

Probabilistic Stress Solution

- Analytic product, for each stress component,
  \[ \Delta \sigma_{ij} = E_{ijkl}^{EP} \Delta \epsilon_{kl} \]

- Incremental stress: each Polynomial Chaos component is updated incrementally
  \[
  \Delta \sigma_{1}^{n+1} = \frac{1}{\langle \Phi_{1} \Phi_{1} \rangle} \{ \sum_{i=1}^{P_{h}} \sum_{k=1}^{P_{e}} h_i \Delta \epsilon_{k}^{n} < \Phi_i \Phi_k \Phi_1 > \}
  - \sum_{j=1}^{P_{g}} \sum_{k=1}^{P_{e}} \sum_{l=1}^{P_{\sigma}} c_j \Delta \epsilon_{k}^{n} \sigma_{l}^{n} < \Phi_j \Phi_k \Phi_l \Phi_1 > \}
  \]

  : 

  \[
  \Delta \sigma_{P}^{n+1} = \frac{1}{\langle \Phi_P \Phi_P \rangle} \{ \sum_{i=1}^{P_{h}} \sum_{k=1}^{P_{e}} h_i \Delta \epsilon_{k}^{n} < \Phi_i \Phi_k \Phi_P > \}
  - \sum_{j=1}^{P_{g}} \sum_{k=1}^{P_{e}} \sum_{l=1}^{P_{\sigma}} c_j \Delta \epsilon_{k}^{n} \sigma_{l}^{n} < \Phi_j \Phi_k \Phi_l \Phi_P > \}
  \]

- Stress update:
  \[
  \sum_{i=1}^{P_{\sigma}} \sigma_{i}^{n+1} \Phi_i = \sum_{i=1}^{P_{\sigma}} \sigma_{i}^{n} \Phi_i + \sum_{i=1}^{P_{\sigma}} \Delta \sigma_{i}^{n+1} \Phi_i
  \]
Probabilistic Elastic-Plastic Modeling
Stochastic Elastic-Plastic Finite Element Method (SEPFEM)

Material uncertainty expanded along stochastic shape functions: 
\[ E(x, t, \theta) = \sum_{i=0}^{P_d} r_i(x, t) \Phi_i[\{\xi_1, ..., \xi_m\}] \]

Loading uncertainty expanded along stochastic shape functions: 
\[ f(x, t, \theta) = \sum_{i=0}^{P_f} f_i(x, t) \zeta_i[\{\xi_{m+1}, ..., \xi_f\}] \]

Displacement expanded along stochastic shape functions: 
\[ u(x, t, \theta) = \sum_{i=0}^{P_u} u_i(x, t) \Psi_i[\{\xi_1, ..., \xi_m, \xi_{m+1}, ..., \xi_f\}] \]
SEPFEM: Formulation

- Stochastic system of equation resulting from Galerkin approach (static example):

\[
\begin{bmatrix}
\sum_{k=0}^{P_d} < \phi_k \psi_0 \psi_0 > K^{(k)} & \cdots & \sum_{k=0}^{P_d} < \phi_k \psi_P \psi_0 > K^{(k)} \\
\sum_{k=0}^{P_d} < \phi_k \psi_0 \psi_1 > K^{(k)} & \cdots & \sum_{k=0}^{P_d} < \phi_k \psi_P \psi_1 > K^{(k)} \\
\vdots & \ddots & \vdots \\
\sum_{k=0}^{P_d} < \phi_k \psi_0 \psi_P > K^{(k)} & \cdots & \sum_{k=0}^{M} < \phi_k \psi_P \psi_P > K^{(k)}
\end{bmatrix}
\begin{bmatrix}
\Delta u_{10} \\
\vdots \\
\Delta u_{NP_u}
\end{bmatrix}
= 
\begin{bmatrix}
\sum_{i=0}^{P_f} f_i < \psi_0 \zeta_i > \\
\sum_{i=0}^{P_f} f_i < \psi_1 \zeta_i > \\
\sum_{i=0}^{P_f} f_i < \psi_2 \zeta_i > \\
\vdots \\
\sum_{i=0}^{P_f} f_i < \psi_P \zeta_i >
\end{bmatrix}
\]

- Time domain integration using Newmark and/or HHT, in probabilistic spaces
SEPFEM: System Size

- SEPFEM offers a complete solution (single step)
- It is not based on Monte Carlo approach
- System of equations does grow (!)

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<th># KL terms load</th>
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</tr>
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SEPFEM: Example in 1D
SEPFEM: Example in 3D

Stochastic Motion of an 8-node brick with Uncertain Young's Modulus

Jeremić and Lacour

SEPFEM
Application of SEPFEM to Practical Problems

Obtain accurate fragility curves (CDFs) for each soil structure system location for stress, strain, displacements, etc.
Summary

Probable importance of uncertainty in mechanics

Propagation of uncertainty through mechanics in order to give designers, regulators and users information for decision making

Real ESSI Simulator: http://real-essi.us

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