Stochastic Elastic-Plastic Finite Element Method: Recent Advances and Developments

Boris Jeremić and Maxime Lacour

University of California, Davis, CA
Lawrence Berkeley National Laboratory, Berkeley, CA

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Крагујевац, Србија

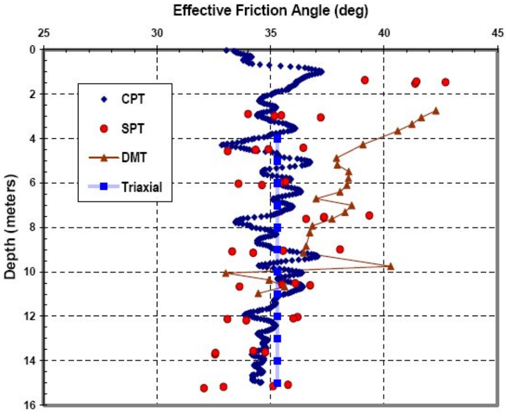
Outline

Motivation

- Probabilistic fish counting
- Williams' DEM simulations, differential displacement vortices
- SFEM round table
- Kavvas' probabilistic hydrology
- Performance based design, probability of undesirable performance, (10^{-4}, 10^{-5} !?)

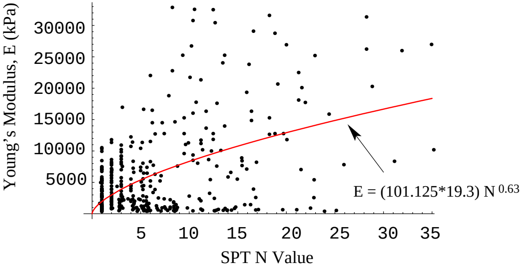
Material Behavior Inherently Uncertain

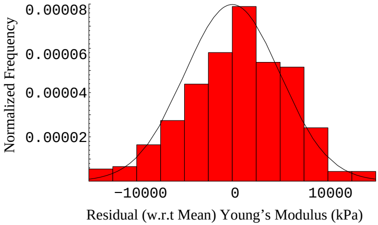
- Spatial variability
- Point-wise uncertainty, testing error, transformation error



(Mayne et al. (2000)

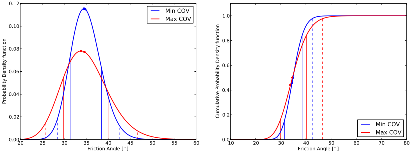
Parametric Uncertainty: Material and Loads



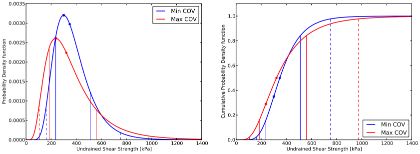


Transformation of SPT *N*-value: 1-D Young's modulus, *E* (cf. Phoon and Kulhawy (1999B))

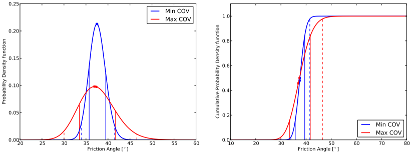
Parametric Uncertainty: Material Properties



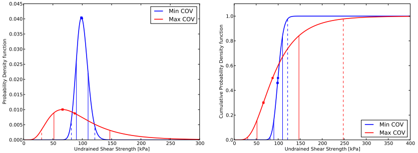
Field ϕ



Field c_u



Lab ϕ



Lab c_u

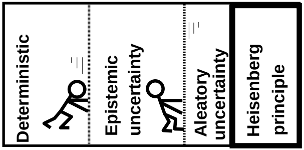
It is not a new Problem

> Le doute n'est pas un état bien agréable,
> mais l'assurance est un état ridicule.

François-Marie Arouet (Voltaire)

Types of Uncertainties

- Epistemic uncertainty - due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools are not well developed
  - trade-off with aleatory uncertainty
- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools
- Ergodic Assumption!?



Outline

Introduction

Probabilistic Inelastic Modeling
FPK Formulations
Direct Solution for Probabilistic Stiffness and Stress

Summary

Uncertainty Propagation through Inelastic System

- Incremental el–pl constitutive equation

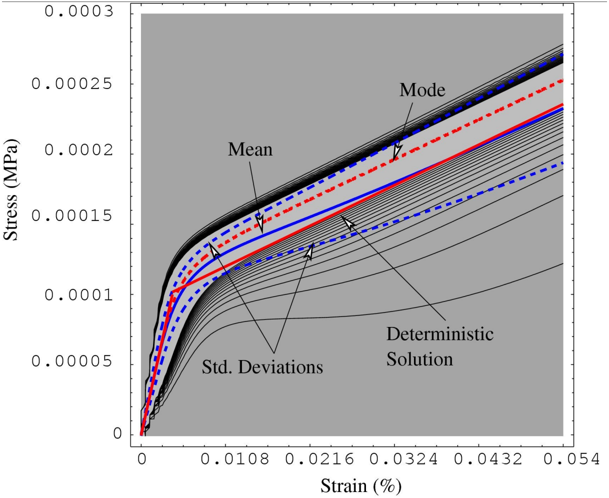
$$\Delta\sigma_{ij} = E^{EP}_{ijkl}\ \Delta\epsilon_{kl} = \left[E^{el}_{ijkl} - \frac{E^{el}_{ijmn} m_{mn} n_{pq} E^{el}_{pqkl}}{n_{rs} E^{el}_{rstu} m_{tu} - \xi_* h_*} \right] \Delta\epsilon_{kl}$$

- Dynamic Finite Elements

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}^{ep}\mathbf{u} = \mathbf{F}$$

- What if all (any) material and load parameters are uncertain

Probabilistic Elastic-Plastic Response



Previous Work

- Linear algebraic or differential equations:
  - Variable Transf. Method (Montgomery and Runger 2003)
  - Cumulant Expansion Method (Gardiner 2004)
- Nonlinear differential equations:
  - Monte Carlo Simulation (Schueller 1997, De Lima et al 2001, Mellah et al. 2000, Griffiths et al. 2005...)
    $\rightarrow$ can be accurate, very costly
  - Perturbation Method (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 1997)
    $\rightarrow$ first and second order Taylor series expansion about mean - limited to problems with small C.O.V. and inherits "closure problem"
  - SFEM (Matthies et al, 2004, 2005, 2014...)

3D FPK Equation

$$
\begin{aligned}
\frac{\partial P(\sigma_{ij}(x_t,t),t)}{\partial t} &= \frac{\partial}{\partial \sigma_{mn}} \left[\left\{ \left\langle \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)) \right\rangle \right.\right. \\
&+ \int_0^t d\tau\, Cov_0 \left[\frac{\partial \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t))}{\partial \sigma_{ab}} ; \right. \\
&\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right] \\
&+ \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[\left\{ \int_0^t d\tau\, Cov_0 \left[\eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)); \right.\right.\right. \\
&\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right]
\end{aligned}
$$

FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right]$$

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- It is deterministic equation in probability density space
- It is linear PDE in probability density space $\rightarrow$ simplifies the numerical solution process

Template Solution of FPK Equation

- FPK diffusion–advection equation is applicable to any material model $\rightarrow$ only the coefficients $N_{(1)}$ and $N_{(2)}$ are different for different material models
- Initial condition
  - Deterministic $\rightarrow$ Dirac delta function $\rightarrow$ $P(\sigma, 0) = \delta(\sigma)$
  - Random $\rightarrow$ Any given distribution
- Boundary condition: Reflecting BC $\rightarrow$ conserves probability mass $\zeta(\sigma, t)|_{At\ Boundaries} = 0$
- Solve using finite differences and/or finite elements
- However (!!) it is a stress solution and probabilistic stiffness is an approximation!

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Probabilistic Inelastic Modeling
- FPK Formulations
- Direct Solution for Probabilistic Stiffness and Stress

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Direct Probabilistic Constitutive Modeling

- Zero elastic region elasto-plasticity with stochastic Armstrong-Frederick kinematic hardening
  $\Delta\sigma = H_a \Delta\epsilon - c_r \sigma |\Delta\epsilon|; \quad E_t = d\sigma/d\epsilon = H_a \pm c_r \sigma$
- Uncertain: init. stiff. H_a, shear strength H_a/c_r, strain $\Delta\epsilon$:
  $H_a = \Sigma h_i \Phi_i; \quad C_r = \Sigma c_i \Phi_i; \quad \Delta\epsilon = \Sigma \Delta\epsilon_i \Phi_i$
- Resulting stress and stiffness are also uncertain

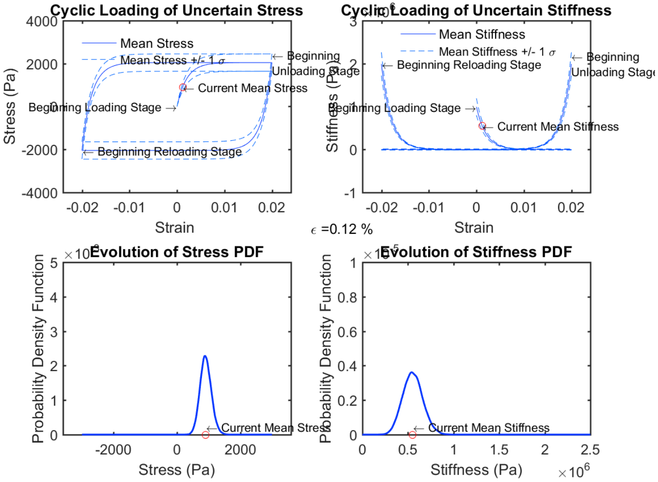
Probabilistic Stiffness Solution

- Analytic product for all the components,
$$E_{ijkl}^{EP} = \left[E_{ijkl}^{el} - \frac{E_{ijmn}^{el} m_{mn} n_{pq} E_{pqkl}^{el}}{n_{rs} E_{rstu}^{el} m_{tu} - \xi_* h_*} \right]$$
- Stiffness: each Polynomial Chaos component is updated incrementally
$$E_{t_1}^{n+1} = \frac{1}{<\Phi_1\Phi_1>}\{\sum_{i=1}^{P_h} h_i < \Phi_i\Phi_1 > \pm \sum_{j=1}^{P_c} \sum_{l=1}^{P_\sigma} c_j \sigma_l^{n+1} < \Phi_j\Phi_l\Phi_1 >\}$$
$$\vdots$$
$$E_{t_P}^{n+1} = \frac{1}{<\Phi_1\Phi_P>}\{\sum_{i=1}^{P_h} h_i < \Phi_i\Phi_P > \pm \sum_{j=1}^{P_c} \sum_{l=1}^{P_\sigma} c_j \sigma_l^{n+1} < \Phi_j\Phi_l\Phi_P >\}$$
- Total stiffness is :
$$E_t^{n+1} = \sum_{l=1}^{P_E} E_{t_i}^{n+1} \Phi_i$$

Probabilistic Stress Solution

- Analytic product, for each stress component,
  $\Delta\sigma_{ij} = E^{EP}_{ijkl}\ \Delta\epsilon_{kl}$
- Incremental stress: each Polynomial Chaos component is updated incrementally
  $\Delta\sigma_1^{n+1} = \frac{1}{<\Phi_1\Phi_1>}\{\sum_{i=1}^{P_h}\sum_{k=1}^{P_e}\ h_i\Delta\epsilon_k^n < \Phi_i\Phi_k\Phi_1 >$
  $-\sum_{j=1}^{P_g}\sum_{k=1}^{P_e}\sum_{l=1}^{P_\sigma}\ c_j\Delta\epsilon_k^n\sigma_l^n < \Phi_j\Phi_k\Phi_l\Phi_1 >\}$
  $\vdots$
  $\Delta\sigma_P^{n+1} = \frac{1}{<\Phi_P\Phi_P>}\{\sum_{i=1}^{P_h}\sum_{k=1}^{P_e}\ h_i\Delta\epsilon_k^n < \Phi_i\Phi_k\Phi_P >$
  $-\sum_{j=1}^{P_g}\sum_{k=1}^{P_e}\sum_{l=1}^{P_\sigma}\ c_j\Delta\epsilon_k^n\sigma_l^n < \Phi_j\Phi_k\Phi_l\Phi_P >\}$
- Stress update:
  $\sum_{l=1}^{P_\sigma}\sigma_i^{n+1}\Phi_i = \sum_{l=1}^{P_\sigma}\sigma_i^n\Phi_i + \sum_{l=1}^{P_\sigma}\Delta\sigma_i^{n+1}\Phi_i$

Probabilistic Elastic-Plastic Modeling



Stochastic Elastic-Plastic Finite Element Method (SEPFEM)

Material uncertainty expanded along stochastic shape functions: $E(x,t,\theta) = \sum_{i=0}^{P_d} r_i(x,t) * \Phi_i[\{\xi_1, ..., \xi_m\}]$

Loading uncertainty expanded along stochastic shape functions: $f(x,t,\theta) = \sum_{i=0}^{P_f} f_i(x,t) * \zeta_i[\{\xi_{m+1}, ..., \xi_f]$

Displacement expanded along stochastic shape functions: $u(x,t,\theta) = \sum_{i=0}^{P_u} u_i(x,t) * \Psi_i[\{\xi_1, ..., \xi_m, \xi_{m+1}, ..., \xi_f\}]$

SEPFEM: Formulation

- Stochastic system of equation resulting from Galerkin approach (static example):

$$\begin{bmatrix} \sum_{k=0}^{P_d} < \Phi_k \Psi_0 \Psi_0 > K^{(k)} & \dots & \sum_{k=0}^{P_d} < \Phi_k \Psi_P \Psi_0 > K^{(k)} \\ \sum_{k=0}^{P_d} < \Phi_k \Psi_0 \Psi_1 > K^{(k)} & \dots & \sum_{k=0}^{P_d} < \Phi_k \Psi_P \Psi_1 > K^{(k)} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=0}^{P_d} < \Phi_k \Psi_0 \Psi_P > K^{(k)} & \dots & \sum_{k=0}^{M} < \Phi_k \Psi_P \Psi_P > K^{(k)} \end{bmatrix} \begin{bmatrix} \Delta u_{10} \\ \vdots \\ \Delta u_{N0} \\ \vdots \\ \Delta u_{1P_u} \\ \vdots \\ \Delta u_{NP_u} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{P_f} f_i < \Psi_0 \zeta_i > \\ \sum_{i=0}^{P_f} f_i < \Psi_1 \zeta_i > \\ \sum_{i=0}^{P_f} f_i < \Psi_2 \zeta_i > \\ \vdots \\ \sum_{i=0}^{P_f} f_i < \Psi_{P_u} \zeta_i > \end{bmatrix}$$

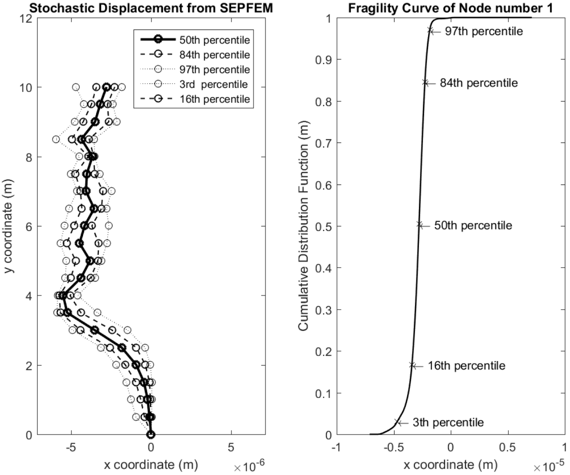
- Time domain integration using Newmark and/or HHT, in probabilistic spaces

SEPFEM: System Size

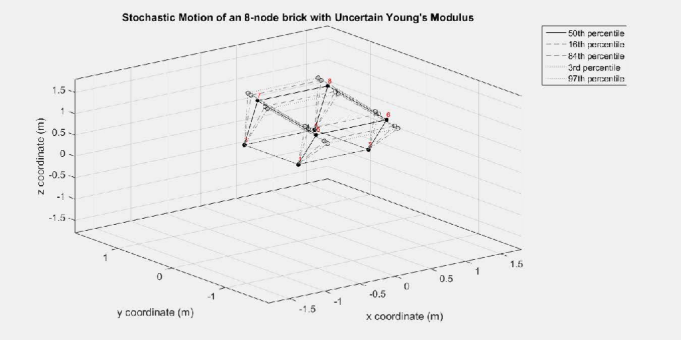
- SEPFEM offers a complete solution (single step)
- It is not based on Monte Carlo approach
- System of equations does grow (!)

# KL terms material	# KL terms load	PC order displacement	Total # terms per DoF
4	4	10	43758
4	4	20	3 108 105
4	4	30	48 903 492
6	6	10	646 646
6	6	20	225 792 840
6	6	30	$1.1058\ 10^{10}$
8	8	10	5 311 735
8	8	20	$7.3079\ 10^{9}$
8	8	30	$9.9149\ 10^{11}$
...	...	...	...

SEPFEM: Example in 1D

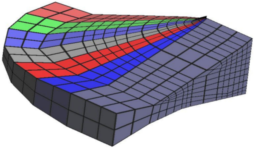


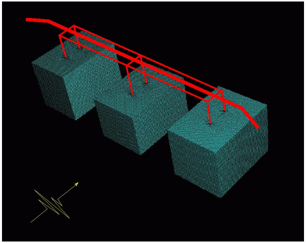
SEPFEM: Example in 3D

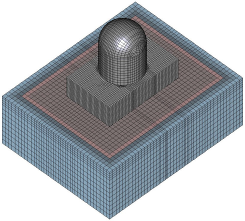


Application of SEPFEM to Practical Problems

Obtain accurate fragility curves (CDFs) for each soil structure system location for stress, strain, displacements, etc.







Summary

Probable importance of uncertainty in mechanics

Propagation of uncertainty through mechanics in order to give designers, regulators and users information for decision making

Real ESSI Simulator: http://real-essi.us

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