Forward and Backward Uncertainty Propagation in Computational Earthquake Engineering

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Outline

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Motivation

Improve modeling and simulation for infrastructure objects

Modeling sophistication level, epistemic uncertainty

Parametric, aleatory uncertainty

Goal: Predict and Inform

Engineer needs to know!

Motivation

François-Marie Arouet, Voltaire: "Le doute n’est pas une condition agréable, mais la certitude est absurde."

Niklaus Wirth: "Software is getting slower more rapidly than hardware becomes faster."

Numerical Prediction under Uncertainty

- Modeling, Epistemic Uncertainty

  Modeling Simplifications

  Modeling sophistication for confidence in results

- Parametric, Aleatory Uncertainty

  $M\ddot{u}_i + C\dot{u}_i + K^{ep} u_i = F(t),$

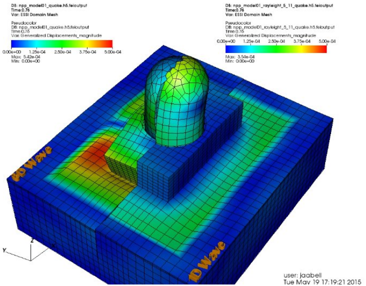
  Uncertain: mass M, viscous damping C and stiffness K^{ep}

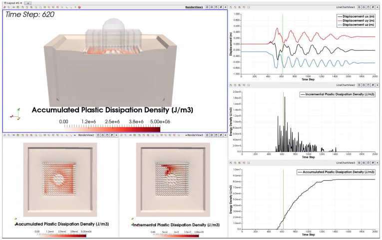
  Uncertain loads, $F(t)$

  Results are PDFs and CDFs for σ_{ij}, ϵ_{ij}, u_i, $\dot{u}_i$, $\ddot{u}_i$

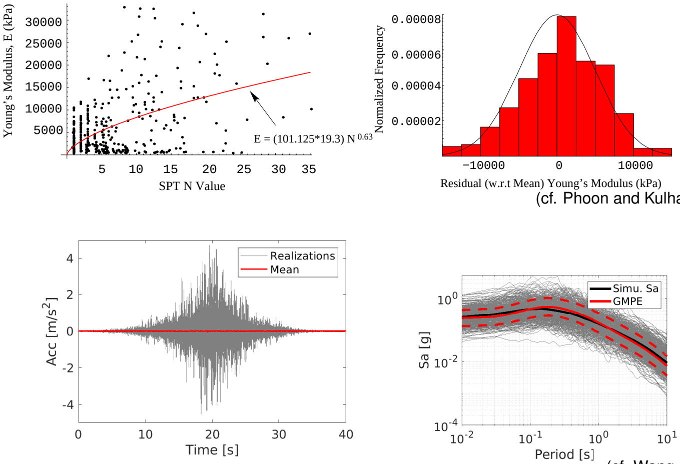
Modeling, Epistemic Uncertainty

- Important (?!) features are simplified, 1C vs 3C, inelasticity
- Modeling simplifications are justifiable if one or two level higher sophistication model demonstrates that features being simplified out are less or not important





Parametric, Aleatory Uncertainty



(cf. Phoon and Kulhawy (1999B))

(cf. Wang et al. (2019))

Engineer Needs to Know!

- Forward propagation of uncertainty, full probabilistic, nonlinear/inelastic Earthquake-Soil-Structure-Interaction, ESSI response in time domain (Jeremic et al 2011, Wang et al 2019)

- Backward propagation, sensitivity analysis, quantifies the relative importance of input uncertain parameters on the variance of the probabilistic system response (Sobol 2001, Sudret 2008, Jeremic et al 2021)

Outline

Forward Uncertain Inelasticity

- Incremental el–pl constitutive equation

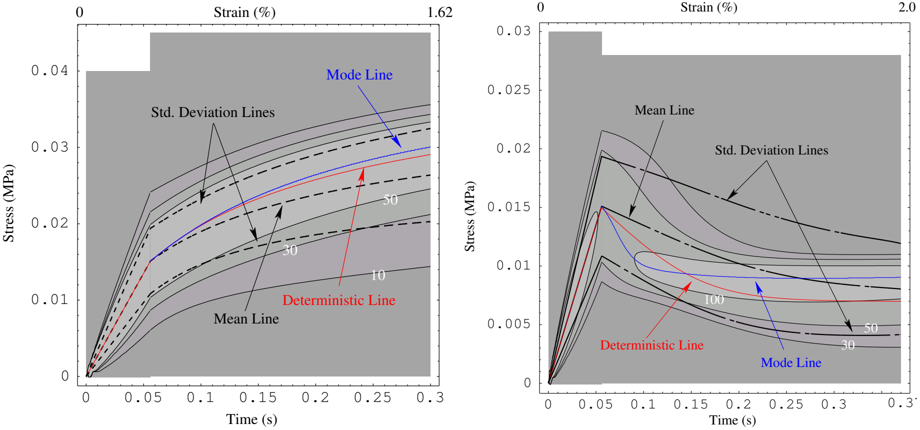
$$\Delta\sigma_{ij} = E^{EP}_{ijkl}\ \Delta\epsilon_{kl} = \left[E^{el}_{ijkl} - \frac{E^{el}_{ijmn} m_{mn} n_{pq} E^{el}_{pqkl}}{n_{rs} E^{el}_{rstu} m_{tu} - \xi_* h_*} \right] \Delta\epsilon_{kl}$$

- Dynamic Finite Elements

$$M\ddot{u}_i + C\dot{u}_i + K^{ep} u_i = F(t)$$

- Material and loads are uncertain

Cam Clay with Random G, M and p_0



Time Domain Stochastic Galerkin Method

Dynamic Finite Elements $M\ddot{u}_i + C\dot{u}_i + K^{ep}u_i = F(t)$

- Input random field/process(non-Gaussian, heterogeneous/ non-stationary): Multi-dimensional Hermite Polynomial Chaos (PC) with known coefficients
- Output response process: Multi-dimensional Hermite PC with unknown coefficients
- Galerkin projection: minimize the error to compute unknown coefficients of response process

Polynomial Chaos Representation

Material random field:
$D(x,\theta) = \sum_{i=1}^{P1} a_i(x)\Psi_i(\{\xi_r(\theta)\})$

Seismic loads/motions random process:
$f_m(t,\theta) = \sum_{j=1}^{P_2} f_{mj}(t)\Psi_j(\{\xi_k(\theta)\})$

Displacement response:
$u_n(t,\theta) = \sum_{k=1}^{P_3} d_{nk}(t)\Psi_k(\{\xi_l(\theta)\})$

where $a_i(x), f_{mj}(t)$ are known PC coefficients, while $d_{nk}(t)$ are unknown PC coefficients.

Direct Probabilistic Constitutive Solution in 1D

- Zero elastic region elasto-plasticity with stochastic Armstrong-Frederick kinematic hardening
  $\Delta\sigma = H_a \Delta\epsilon - c_r \sigma |\Delta\epsilon|; \quad E_t = d\sigma / d\epsilon = H_a \pm c_r \sigma$
- Uncertain: init. stiff. H_a, shear strength H_a / c_r, strain $\Delta\epsilon$:
  $H_a = \Sigma h_i \Phi_i; \quad C_r = \Sigma c_i \Phi_i; \quad \Delta\epsilon = \Sigma \Delta\epsilon_i \Phi_i$
- Resulting stress and stiffness are also uncertain

Direct Probabilistic Stiffness Solution

- Analytic product for all the components,
  $$E_{ijkl}^{EP} = \left[E_{ijkl}^{el} - \frac{E_{ijmn}^{el} m_{mn} n_{pq} E_{pqkl}^{el}}{n_{rs} E_{rstu}^{el} m_{tu} - \xi_* h_*} \right]$$
- Stiffness: each Polynomial Chaos component is updated incrementally
  $E_{t_1}^{n+1} = \frac{1}{<\Phi_1\Phi_1>}\{\sum_{i=1}^{P_h}\ h_i < \Phi_i\Phi_1 > \pm \sum_{j=1}^{P_c}\sum_{l=1}^{P_\sigma}\ c_j\sigma_l^{n+1} < \Phi_j\Phi_l\Phi_1 >\}$
  . . .
  $E_{t_P}^{n+1} = \frac{1}{<\Phi_1\Phi_P>}\{\sum_{i=1}^{P_h}\ h_i < \Phi_i\Phi_P > \pm \sum_{j=1}^{P_c}\sum_{l=1}^{P_\sigma}\ c_j\sigma_l^{n+1} < \Phi_j\Phi_l\Phi_P >\}$
- Total stiffness is :
  $E_t^{n+1} = \sum_{l=1}^{P_E} E_{t_i}^{n+1}\Phi_i$

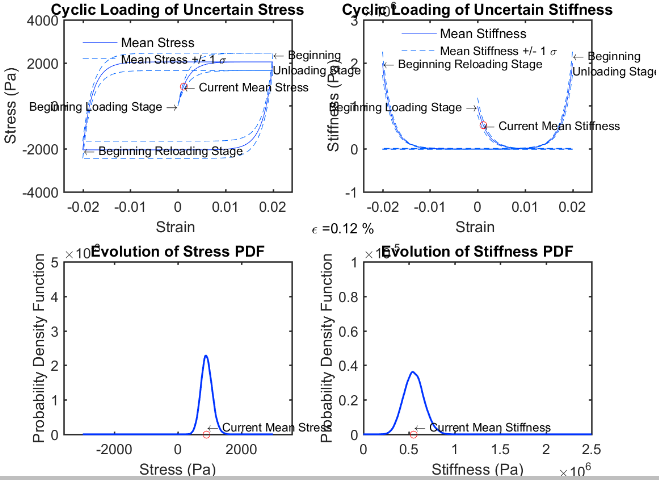
Direct Probabilistic Stress Solution

- Analytic product, for each stress component,
  $\Delta\sigma_{ij} = E^{EP}_{ijkl}\ \Delta\epsilon_{kl}$

- Incremental stress: each Polynomial Chaos component is updated incrementally
  $\Delta\sigma^{n+1}_{1} = \frac{1}{<\Phi_1\Phi_1>}\{\sum_{i=1}^{P_h}\sum_{k=1}^{P_e}\ h_i\Delta\epsilon^n_k < \Phi_i\Phi_k\Phi_1 >$
  $-\sum_{j=1}^{P_g}\sum_{k=1}^{P_e}\sum_{l=1}^{P_\sigma}\ c_j\Delta\epsilon^n_k\sigma^n_l < \Phi_j\Phi_k\Phi_l\Phi_1 >\}$
  . . .
  $\Delta\sigma^{n+1}_{P} = \frac{1}{<\Phi_P\Phi_P>}\{\sum_{i=1}^{P_h}\sum_{k=1}^{P_e}\ h_i\Delta\epsilon^n_k < \Phi_i\Phi_k\Phi_P >$
  $-\sum_{j=1}^{P_g}\sum_{k=1}^{P_e}\sum_{l=1}^{P_\sigma}\ c_j\Delta\epsilon^n_k\sigma^n_l < \Phi_j\Phi_k\Phi_l\Phi_P >\}$

- Stress update:
  $\sum_{l=1}^{P_\sigma}\sigma^{n+1}_i\Phi_i = \sum_{l=1}^{P_\sigma}\sigma^n_i\Phi_i + \sum_{l=1}^{P_\sigma}\Delta\sigma^{n+1}_i\Phi_i$

Probabilistic Elastic-Plastic Response



(MP4)

Stochastic Elastic-Plastic Finite Element Method

- Material uncertainty expanded into stochastic shape funcs.
- Loading uncertainty expanded into stochastic shape funcs.
- Displacement expanded into stochastic shape funcs.
- Jeremić et al. 2011

$$
\begin{bmatrix}
\sum_{k=0}^{P_d} < \Phi_k \Psi_0 \Psi_0 > K^{(k)} & \dots & \sum_{k=0}^{P_d} < \Phi_k \Psi_P \Psi_0 > K^{(k)} \\
\sum_{k=0}^{P_d} < \Phi_k \Psi_0 \Psi_1 > K^{(k)} & \dots & \sum_{k=0}^{P_d} < \Phi_k \Psi_P \Psi_1 > K^{(k)} \\
\vdots & \vdots & \vdots \\
\sum_{k=0}^{P_d} < \Phi_k \Psi_0 \Psi_P > K^{(k)} & \dots & \sum_{k=0}^{M} < \Phi_k \Psi_P \Psi_P > K^{(k)}
\end{bmatrix}
\begin{bmatrix}
\Delta u_{10} \\ \vdots \\ \Delta u_{N0} \\ \vdots \\ \Delta u_{1P_u} \\ \vdots \\ \Delta u_{NP_u}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{i=0}^{P_f} f_i < \Psi_0 \zeta_i > \\
\sum_{i=0}^{P_f} f_i < \Psi_1 \zeta_i > \\
\sum_{i=0}^{P_f} f_i < \Psi_2 \zeta_i > \\
\vdots \\
\sum_{i=0}^{P_f} f_i < \Psi_{P_u} \zeta_i >
\end{bmatrix}
$$

SEPFEM: System Size

- SEPFEM offers a complete probabilistic solution
- It is NOT based on Monte Carlo approach
- System of equations grows (!)

# KL terms material	# KL terms load	PC order displacement	Total # terms per DoF
4	4	10	43758
4	4	20	3 108 105
4	4	30	48 903 492
6	6	10	646 646
6	6	20	225 792 840
6	6	30	$1.1058\ 10^{10}$
8	8	10	5 311 735
8	8	20	$7.3079\ 10^{9}$
8	8	30	$9.9149\ 10^{11}$
...	...	...	...

Outline

ANOVA Representation

Model with n uncertain inputs ($\boldsymbol{x}$) and scalar output y:

$$y = f(\boldsymbol{x}); \;\; \boldsymbol{x} \in I^n$$

The ANalysis Of VAriance representation (Sobol 2001):

$$f(x_1, ...x_n) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{1 \leq i < j \leq n} f_{ij}(x_i, x_j) + ...f_{1,...n}(x_1, ...x_n)$$

Sensitivity Analysis, ANOVA Representation

Total of 2^n summands

Mean value $f_0 = \int_{I^n} f(\boldsymbol{x}) d\boldsymbol{x}$

Integral of each summand $\int_0^1 f_{i_1,...i_s}(x_{i_1}, ...x_{i_s}) dx_{i_k} = 0; 1 \leq k \leq s$

Summands are orthogonal to each other

$$\int_{I^n} f_{i_1,...i_s}(x_{i_1}, x_{i_2}, ...x_{i_s}) f_{j_1,...j_t}(x_{j_1}, x_{j_2}, ...x_{j_t}) d\boldsymbol{x} = 0;$$

$$\{i_1, ...i_s\} \neq \{j_1, ...j_t\}$$

Sensitivity Analysis, ANOVA Representation

ANOVA representation is unique!

Univariate terms:

$$f_i(x_i) = \int_{I^{n-1}} f(\boldsymbol{x})\, d\boldsymbol{x}_{\sim i} - f_0$$

Bivariate terms:

$$f_{ij}(x_i, x_j) = \int_{I^{n-2}} f(\boldsymbol{x})\, d\boldsymbol{x}_{\sim [ij]} - f_i(x_i) - f_j(x_j) - f_0$$

Sensitivity Analysis, Variance

Total variance of the probabilistic model response $y = f(\boldsymbol{X})$ is

$$D = \mathrm{Var}[f(\boldsymbol{X})] = \int_{I^n} f^2(\boldsymbol{x})\ d\boldsymbol{x} - f_0^2$$

Total variance D, decomposed

$$D = \sum_{i=1}^{n} D_i + \sum_{1 \le i < j \le n} D_{ij} + ... + D_{1,2,...,n} = \sum_{s=1}^{n} \sum_{i_1 < ... < i_s} D_{i_1 ... i_s}$$

Variance contribution from individual summand

$$D_{i_1 .. i_s} = \int_{I^s} f^2_{i_1 ... i_s}(x_{i_1}, ..., x_{i_s})\ dx_{i_1}, ... dx_{i_s},$$
$$1 \le i_1 < ... < i_s \le n;\ s = 1, ...n$$

Sobol Indices

- Sobol' indices $S_{i_1 \ldots i_s}$, fractional contributions from random inputs $\{X_{i_1}, ..., X_{i_s}\}$ to the total variance D: $S_{i_1 \ldots i_s} = D_{i_1 \ldots i_s}/D$
- First order indices $S_i \rightarrow$ individual influence of each uncertain input parameter
- Higher order indices $S_{i_1 \ldots i_s} \rightarrow$ mixed influence from groups of uncertain input parameters
- Total sensitivity indices, influence of input parameter X_i

$$S_i^{\mathrm{total}} = \sum_{\mathscr{S}_i} D_{i_1 \ldots i_s}$$

Sobol Indices and Polynomial Chaos

PC expansion of response written in ANOVA form (Sudret 2008)

Multi-dimensional PC bases $\{\Psi_j(\boldsymbol{\xi})\}$ decomposed into products of single dimension PC chaos bases of different orders

$$\Psi_j(\boldsymbol{\xi}) = \prod_{i=1}^{n} \phi_{\alpha_i}(\xi_i) \tag{1}$$

$\phi_{\alpha_i}(\xi_i)$ is the single dimensional, order α_i, polynomial function of underlying basic random variable ξ_i.

Sobol Sensitivity Analysis

From ANOVA representation of probabilistic model response, the PC-based Sobol' indices $S^{PC}_{i_1 \ldots i_s}$ are

$$S^{PC}_{i_1 \ldots i_s} = \sum_{\boldsymbol{\alpha} \in \mathscr{S}_{i_1,\ldots,i_s}} y^2_{\boldsymbol{\alpha}} \boldsymbol{E}\left[\Psi^2_{\boldsymbol{\alpha}}\right] / D^{PC}$$

Sobol Sensitivity Analysis

Total Sobol' indices $S^{PC,\text{total}}_{j_1 \dots j_t}$

$$S^{PC,\text{total}}_{j_1 \dots j_t} = \sum_{(i_1,\dots,i_s) \in \mathscr{S}_{j_1,\dots,j_t}} S^{PC}_{i_1 \dots i_s}$$

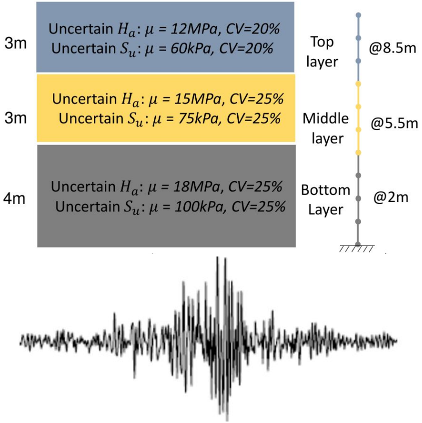
where $\mathscr{S}_{j_1,\dots,j_t} = \{(i_1, \dots, i_s) : (j_1, \dots, j_t) \subset (i_1, \dots, i_s)\}$

Using PC representation of probabilistic model response, Sobol' sensitivity indices are analytic and inexpensive

Outline

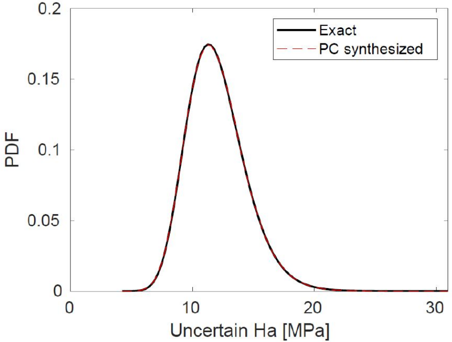
Stochastic Site Response

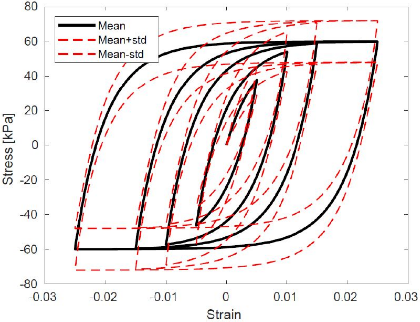
- Uncertain material: uncertain random field, marginally lognormal distribution, exponential correlation length 10m
- Uncertain seismic rock motions: seismic scenario M=7, R=50km



Stochastic Material Parameters

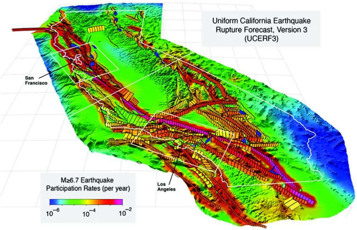
Lognormal distributed random field with PC Dim. 3 Order 2

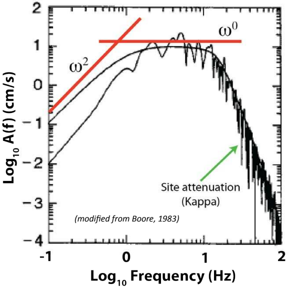




Stochastic Seismic Motion Development

- UCERF3 (Field et al. 2014)
- Stochastic motions (Boore 2003)
- Polynomial Chaos Karhunen-Loève expansion
- Probabilistic DRM (Bielak et al. 2003, Wang et al. 2021)

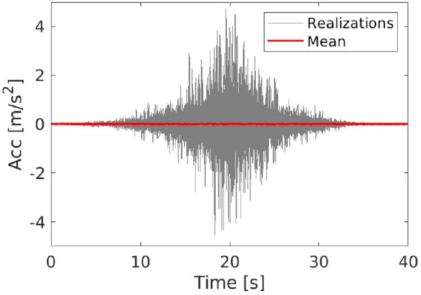


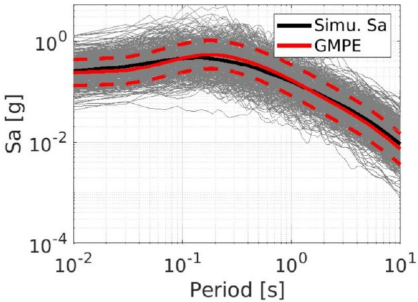


Stochastic Ground Motion Modeling

- Modeling fundamental characteristics of uncertain ground motions, Stochastic Fourier amplitude spectra (FAS). and Stochastic Fourier phase spectra (FPS) and not specific IM
- Mean behavior of stochastic FAS, w^2 source radiation spectrum by Brune(1970), and Boore(1983, 2003, 2015).
- Variability models for stochastic FAS, FAS GMPEs by Bora et al. (2015, 2018), Bayless & Abrahamson (2019), Stafford(2017) and Bayless & Abrahamson (2018).
- Stochastic FPS by phase derivative (Boore,2005), Logistic phase derivative model by Baglio & Abrahamson (2017)

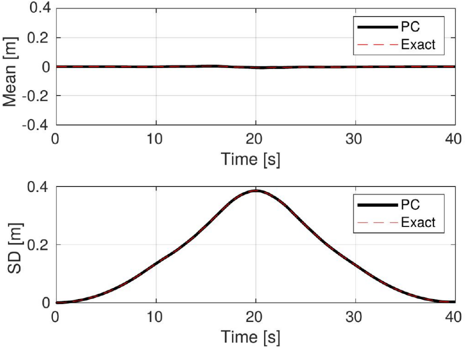
Stochastic Seismic Motions



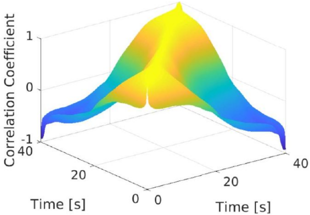


Stochastic Seismic Motions, Displacement

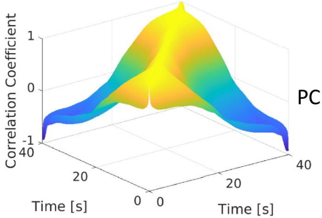
- Displacement: **PC Dim. 150 Order 1**



Marginal behavior



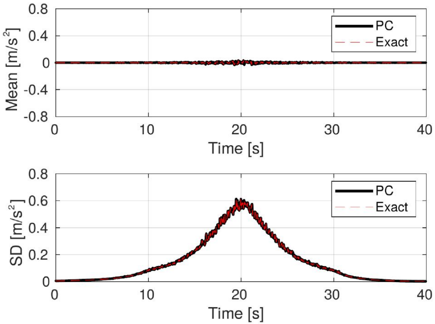
Exact



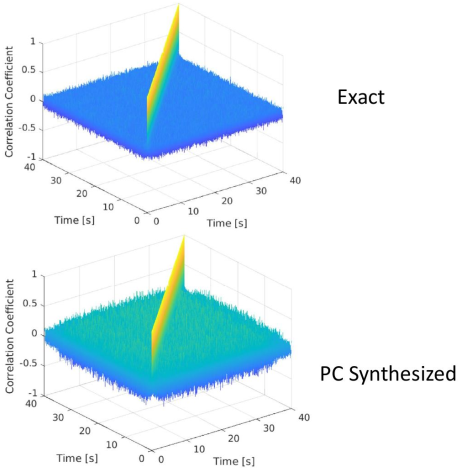
PC Synthesized

Stochastic Seismic Motions, Accelerations

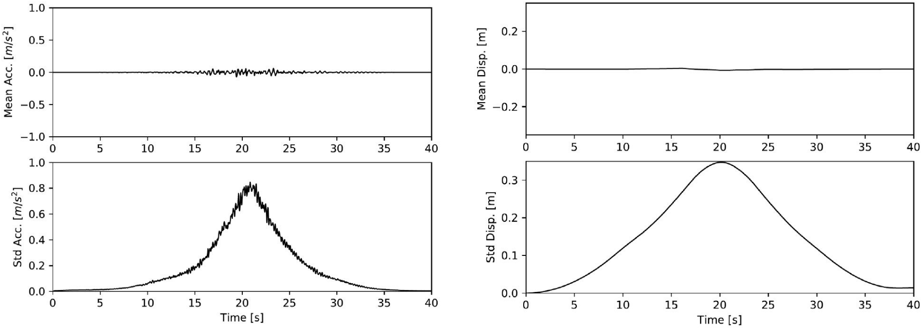
- Acceleration: **PC Dim. 150 Order 1**



Marginal behavior



Stochastic Site Response



Acceleration | Displacement

PC dimensions ξ_1, - ξ_3 are from uncertain soil
PC dimensions ξ_4, - ξ_{153} are from uncertain rock motions

Sensitivity of PGA from Uncertain Soil

- First 10 terms from soil uncertainty
- Total Sobol sensitivity index $S_{1-3}^{PC,total} = 0.51$

Sobol Index	Value
$S_{1,123}^{PC}$	0,04389
$S_{1,118}^{PC}$	0,02605
$S_{1,127}^{PC}$	0,02370
$S_{1,100}^{PC}$	0,01759
$S_{1,103}^{PC}$	0,01700
$S_{1,134}^{PC}$	0,01680
$S_{1,141}^{PC}$	0,01611
$S_{1,110}^{PC}$	0,01358
$S_{1,130}^{PC}$	0,01303
$S_{1,132}^{PC}$	0,01068
...	...

Sensitivity of PGA from Uncertain Rock Motions

- First 10 terms from motions uncertainty
- Total Sobol sensitivity index $S_{1-153}^{PC,total} = 0.98$

Sobol Index	Value
$S_{1,123}^{PC}$	0,04389
S_{137}^{PC}	0,03459
S_{110}^{PC}	0,03061
S_{118}^{PC}	0,02698
$S_{1,118}^{PC}$	0,02605
S_{108}^{PC}	0,02482
S_{141}^{PC}	0,02373
$S_{1,127}^{PC}$	0,02370
$S_{1,100}^{PC}$	0,01759
$S_{1,103}^{PC}$	0,01700
...	...

Sensitivity Analysis

Total variance in PGA, in this case (!), dominated by uncertain ground motions

- 49% from uncertain rock motions at depth
- 2% from uncertain soil
- 49% from interaction of uncertain rock motions and uncertain soil

Outline

Summary

- Analysis of uncertainties and sensitivities
- Predict and inform
- Engineer needs to know!
- `http://real-essi.us/`

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