Probabilistic Elasto-Plasticity

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Introduction

- Material behavior is stochastic, both spatially and pointwise,
- How is failure of solids and structures affected by that stochasticity?
- Use random theory to propagate the effects of stochastic (here made into random) material properties through elastic and elastic–plastic constitutive equations
- Use of Forward Kolmogorov (Fokker–Planck) equation in Eulerian–Lagragian settings
- This is pointwise analysis, current work on spatial extension
- Results represent non–Markovian process

Presentation Overview

- Problem Statement
- Method of Solution: Forward Kolmogorov (Fokker-Planck) Approach
- Examples and Verification
  - Elastic
  - Drucker-Prager Linear Hardening
  - Cam Clay
- Conclusions

Problem Statement

- The general 3-D constitutive rate equation - a nonlinear ODE system with random coefficient and random forcing

$$\frac{d\sigma_{ij}(t)}{dt} = D_{ijkl}\frac{d\epsilon_{kl}(t)}{dt}$$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{when } f < 0 \vee (f = 0 \wedge df < 0) \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn}\dfrac{\partial U}{\partial \sigma_{mn}}\dfrac{\partial f}{\partial \sigma_{pq}}D^{el}_{pqkl}}{\dfrac{\partial f}{\partial \sigma_{rs}}D^{el}_{rstu}\dfrac{\partial U}{\partial \sigma_{tu}} - \dfrac{\partial f}{\partial q_*}r_*} & \text{when } f = 0 \vee df = 0 \end{cases}$$

- 1-D – a nonlinear ODE, random coefficient and random forcing

$$\frac{d\sigma(t)}{dt} = \beta(\sigma, D, q, r; t)\frac{d\epsilon(t)}{dt} = \eta(\sigma, D, q, r, \epsilon; t)$$

with an initial condition $\sigma(0) = \sigma_0$

Stochastic Continuity Equation

- The 1-D constitutive equation visualization: from each initial point in σ-space a trajectory starts out which describes the corresponding solution of the stochastic process

- Consider a cloud of initial points (described by density $\rho(\sigma, 0)$ in σ-space): movement of all these points is dictated by the constitutive equation, the phase density ρ varies in time according to a continuity equation (Liouville equation):

$$\frac{\partial \rho(\sigma(t), t)}{\partial t} = -\frac{\partial}{\partial \sigma} \eta[\sigma(t), D, q, r, \epsilon(t)].\rho[\sigma(t), t]$$

with initial condition

$$\rho(\sigma, 0) = \delta(\sigma - \sigma_0)$$

Fokker-Planck Equation

- Writing the continuity equation in ensemble average form and using Van Kampen's Lemma ($<\rho(h,t)>= P(h,t)$) yields the following Fokker-planck equation:

$$
\begin{aligned}
\frac{\partial P(\sigma(t),t)}{\partial t} = & - \frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(t),D,q,r,\epsilon(t))\right\rangle\right.\right. \\
& + \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(t),D,q,r,\epsilon(t))}{\partial \sigma};\right. \\
& \left.\left.\left.\eta(\sigma(t-\tau),D,q,r,\epsilon(t-\tau)\right]\right\} P(\sigma(t),t)\right] \\
& + \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(t),D,q,r,\epsilon(t));\right.\right.\right. \\
& \left.\left.\left.\eta(\sigma(t-\tau),D,q,r,\epsilon(t-\tau))\right]\right\} P(\sigma(t),t)\right]
\end{aligned}
$$

Solution of Fokker-Planck Equation

- The Fokker-Planck equation $\rightarrow$ advection-diffusion equation:

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition – deterministic (Dirac delta function) or random

$$P(\sigma,0) = \delta(\sigma)$$

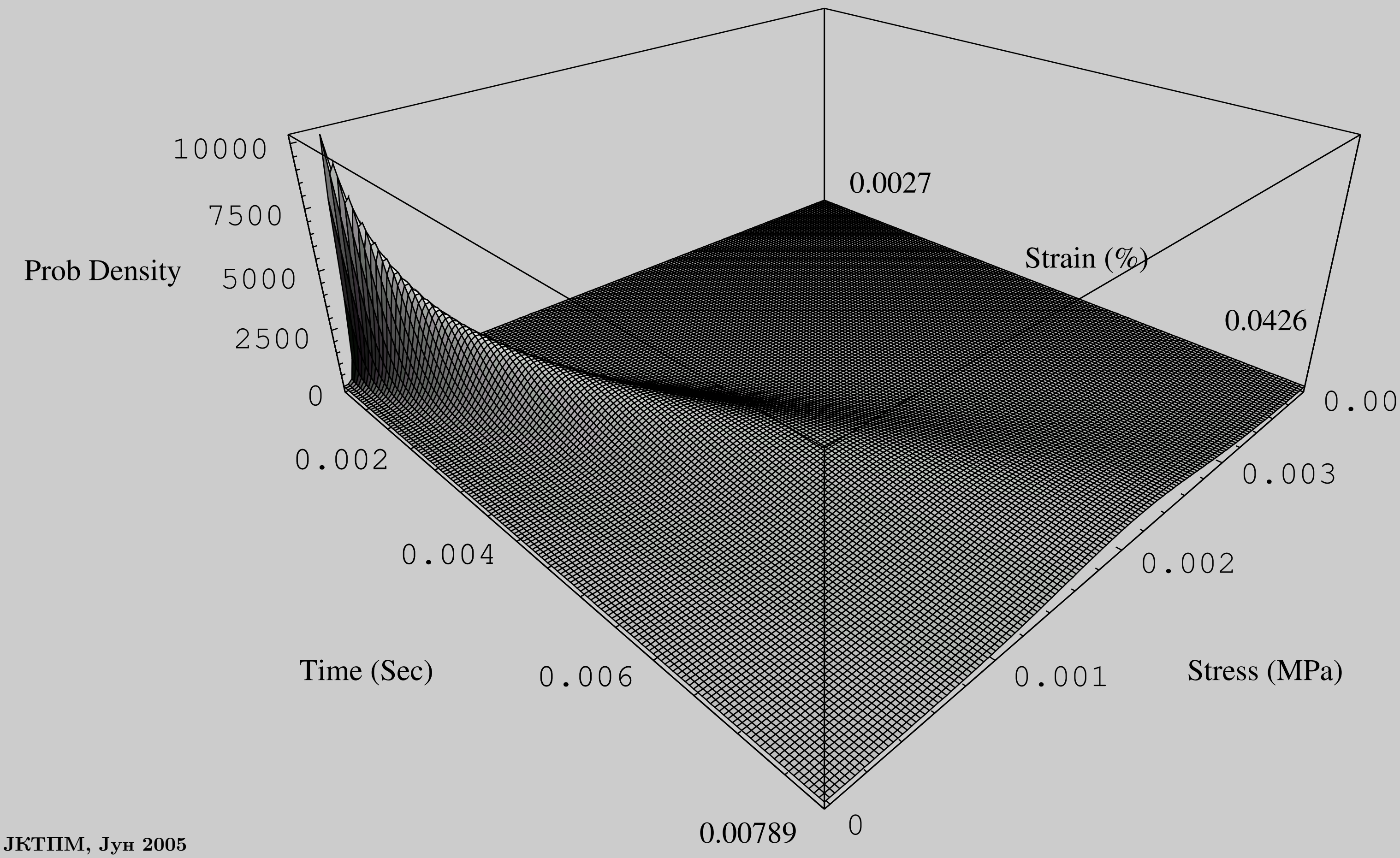
- Boundary condition – reflectind (conserve probability mass or no probability current flow)

$$\zeta(\sigma,t)|_{AtBoundaries} = 0$$

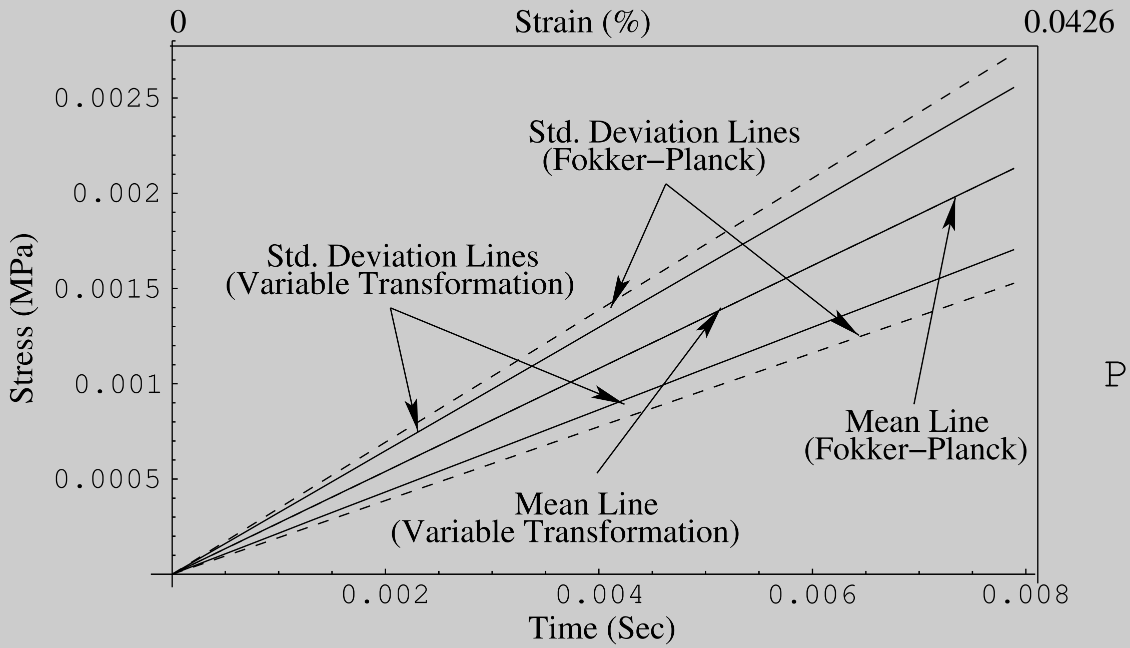
- The Fokker-Planck equation solution $\rightarrow$ *Finite Difference Technique*

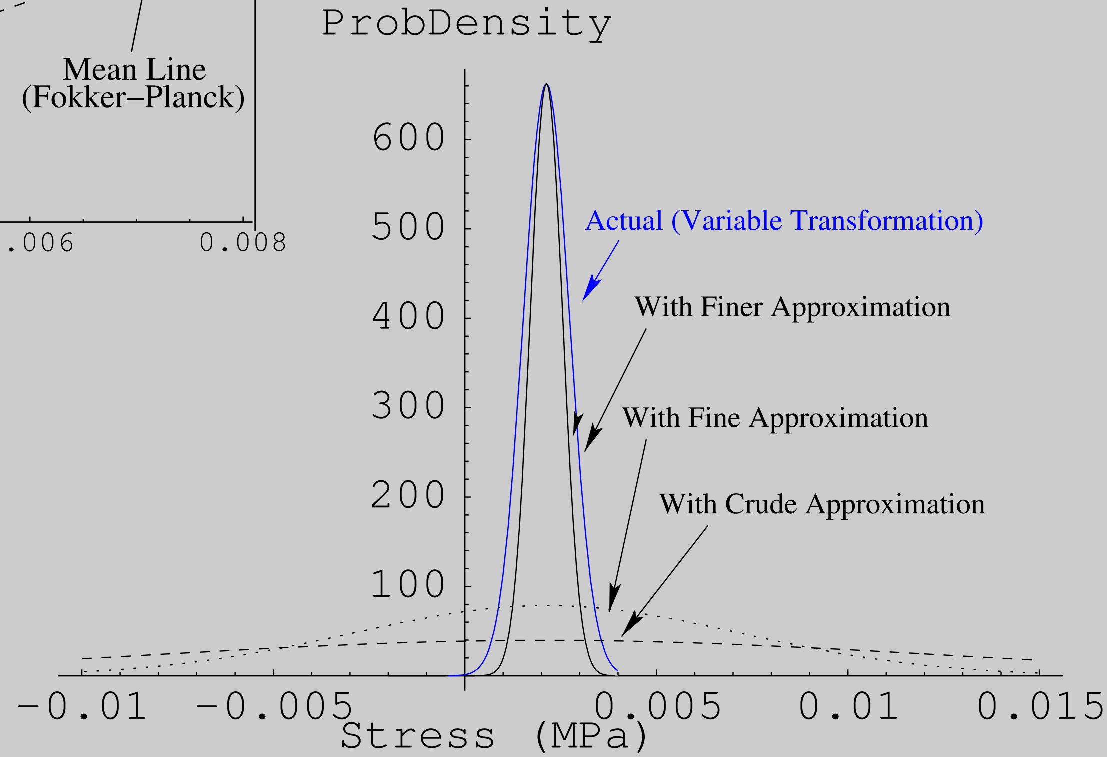
Elastic Response with Random G

- General form of elastic constitutive rate equation
  $d\sigma_{12}/dt = 2G d\epsilon_{12}/dt = \eta(G, \epsilon_{12}; t)$
- The advection and diffusion coefficents of FPE are
  $N_{(1)} = 2d\epsilon_{12}/dt < G > \;\; ; \;\; N_{(2)} = 4t\left(d\epsilon_{12}/dt\right)^2 Var[G]$



Verification of Elastic Response Variable Transformation





Drucker-Prager Associative Linear Hardening with Random G

- The general form of Drucker-Prager elastic-plastic associative linear hardening constitutive rate equation

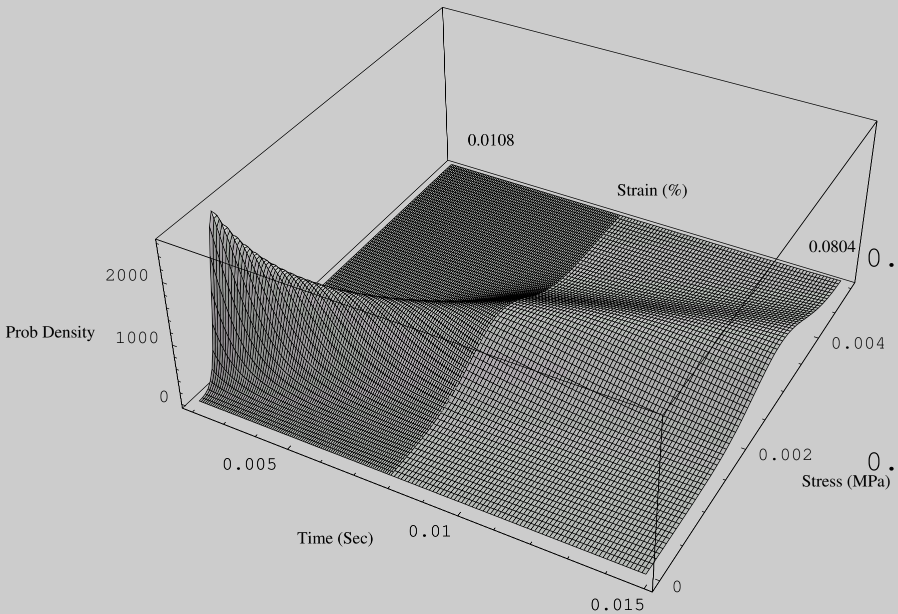
$$\begin{aligned}\frac{d\sigma_{12}}{dt} &= G^{ep}\frac{d\epsilon_{12}}{dt} \\ &= \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)\end{aligned}$$

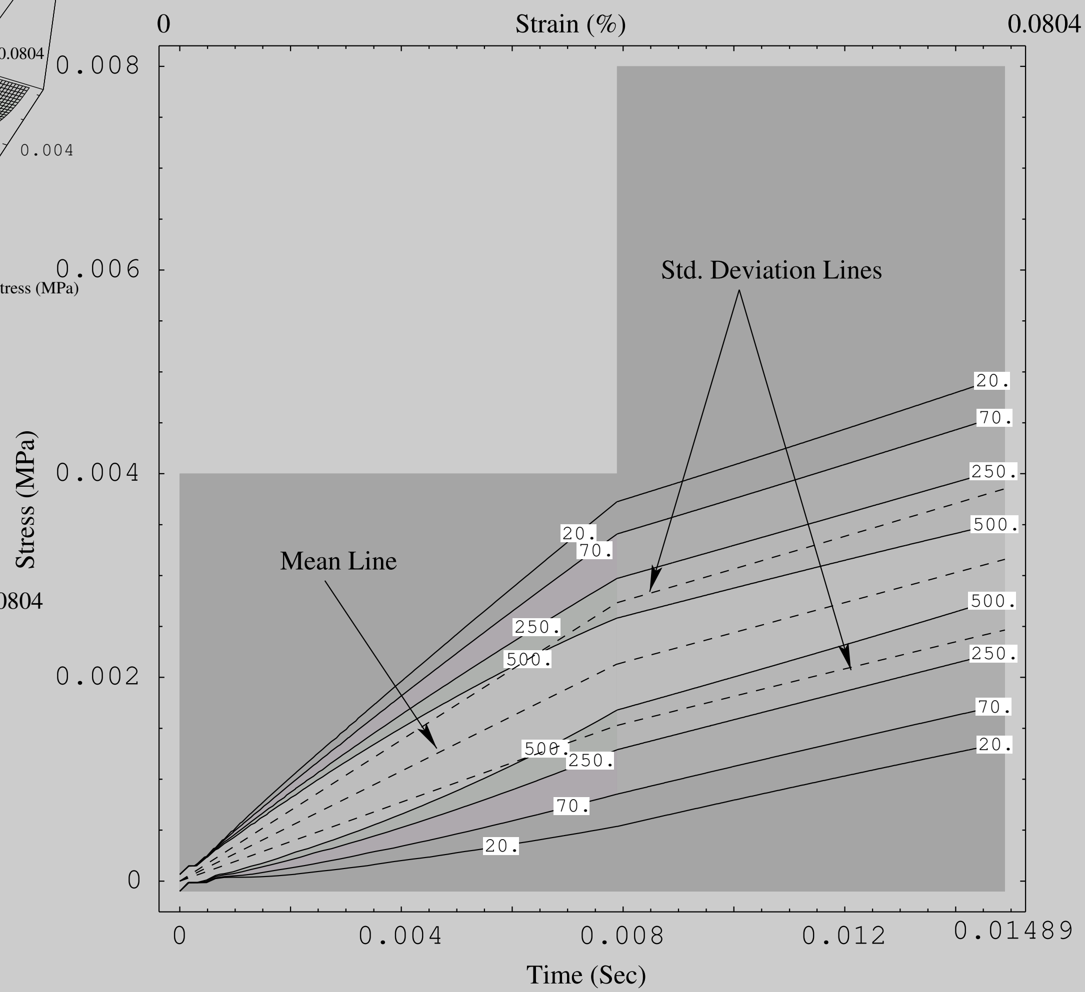
- The advection and diffusion coefficents of FPE are

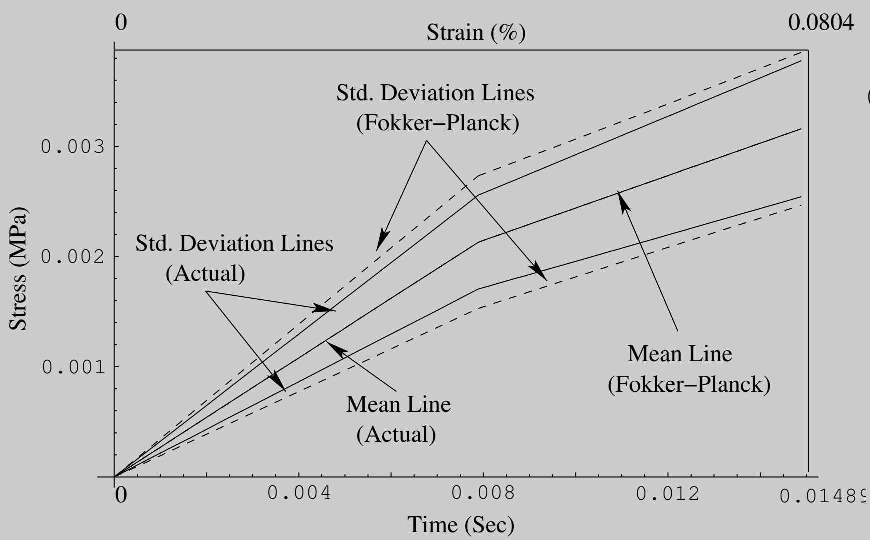
$$N_{(1)} = \frac{d\epsilon_{12}}{dt}\left\langle 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right\rangle$$

$$N_{(2)} = t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var\left[2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right]$$

Drucker-Prager Associative Linear Hardening with Random G







Cam Clay Constitutive Model

- The general form of Cam Clay 1-D shear constitutive rate equation

$$\begin{aligned}\frac{d\sigma_{12}}{dt} &= G^{ep}\frac{d\epsilon_{12}}{dt} \\ &= \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)\end{aligned}$$

where η has the form:

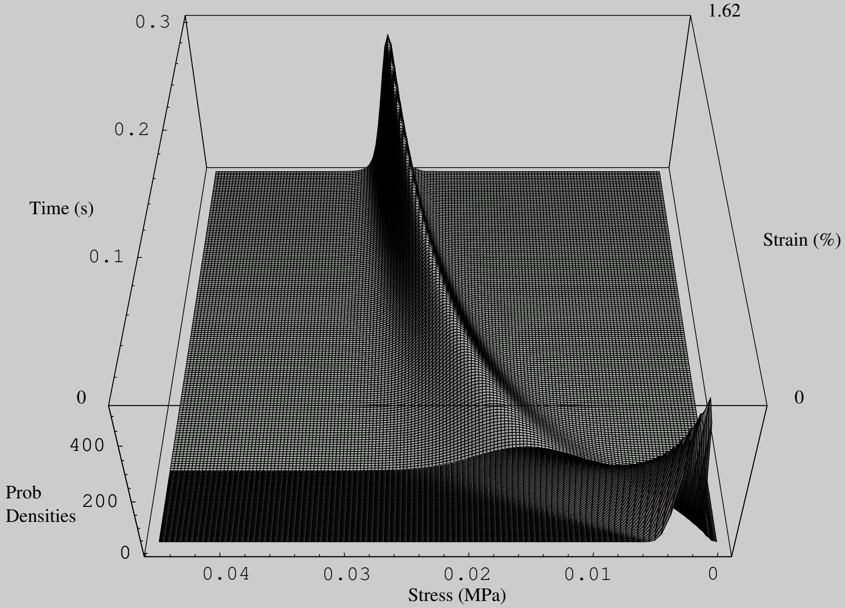
$$\eta = \left[2G - \frac{\left(36\frac{G^2}{M^4}\right)\sigma_{12}^2}{\frac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\frac{G}{M^4}\right)\sigma_{12}^2 + \frac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)}\right]\frac{d\epsilon_{12}}{dt}$$

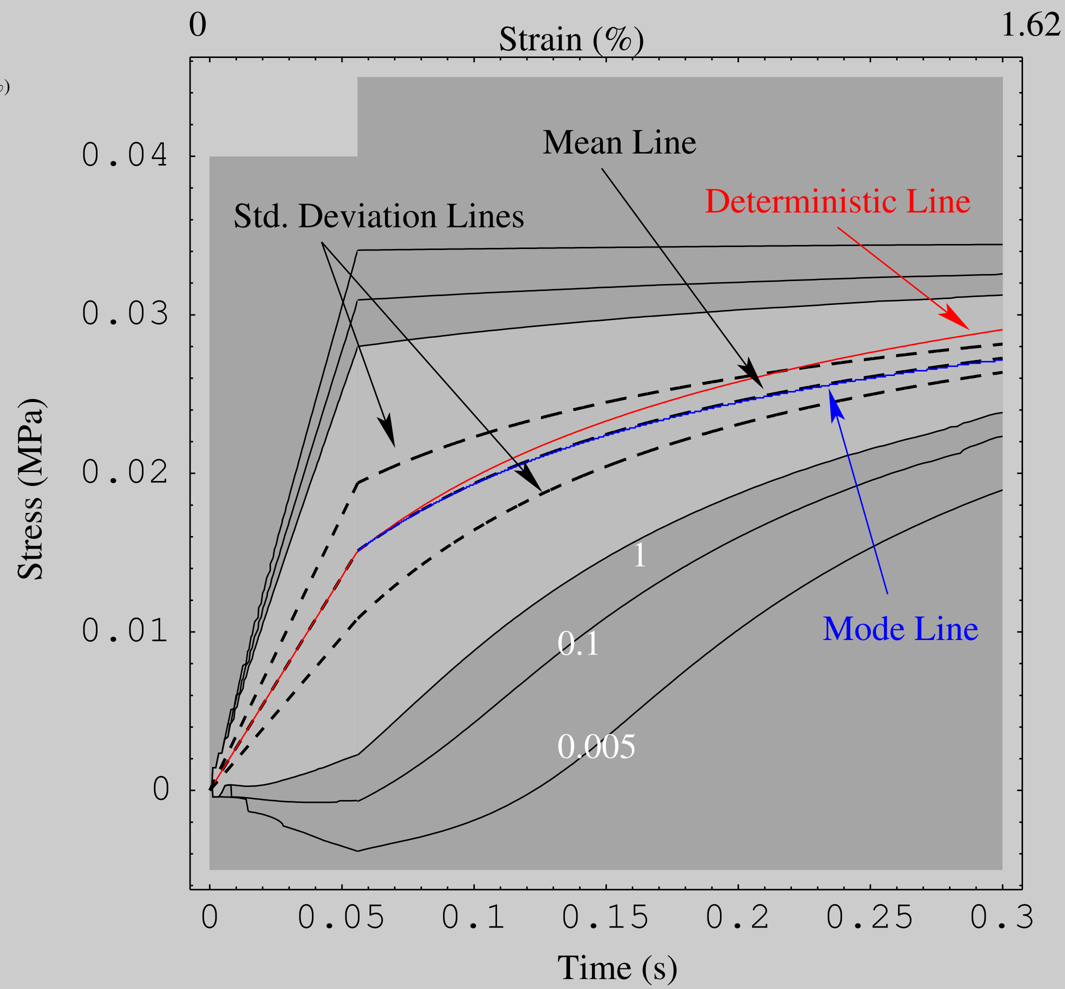
- The advection and diffusion coefficents of FPE are

$$N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov\left[\frac{\partial\eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau)\right]$$

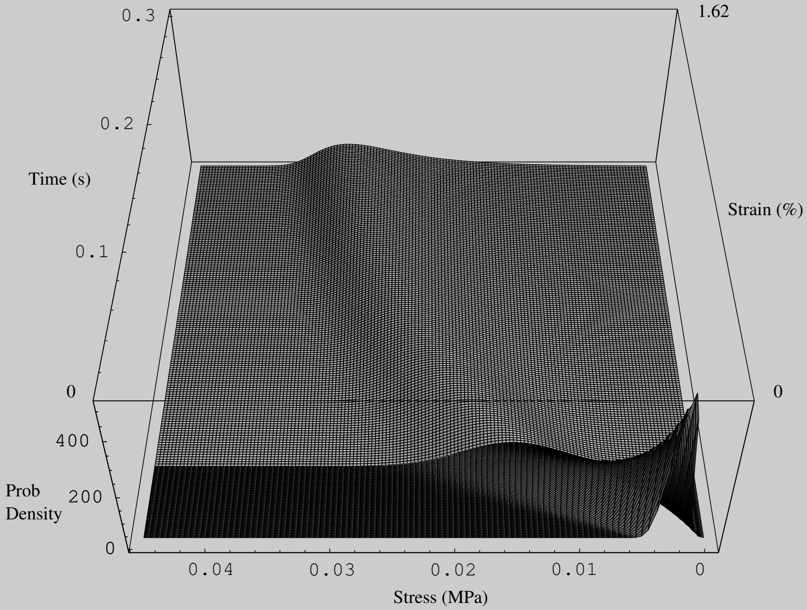
$$N_{(2)}^{(i)} = \int_0^t d\tau cov\left[\eta^{(i)}(t); \eta^{(i)}(t-\tau)\right]$$

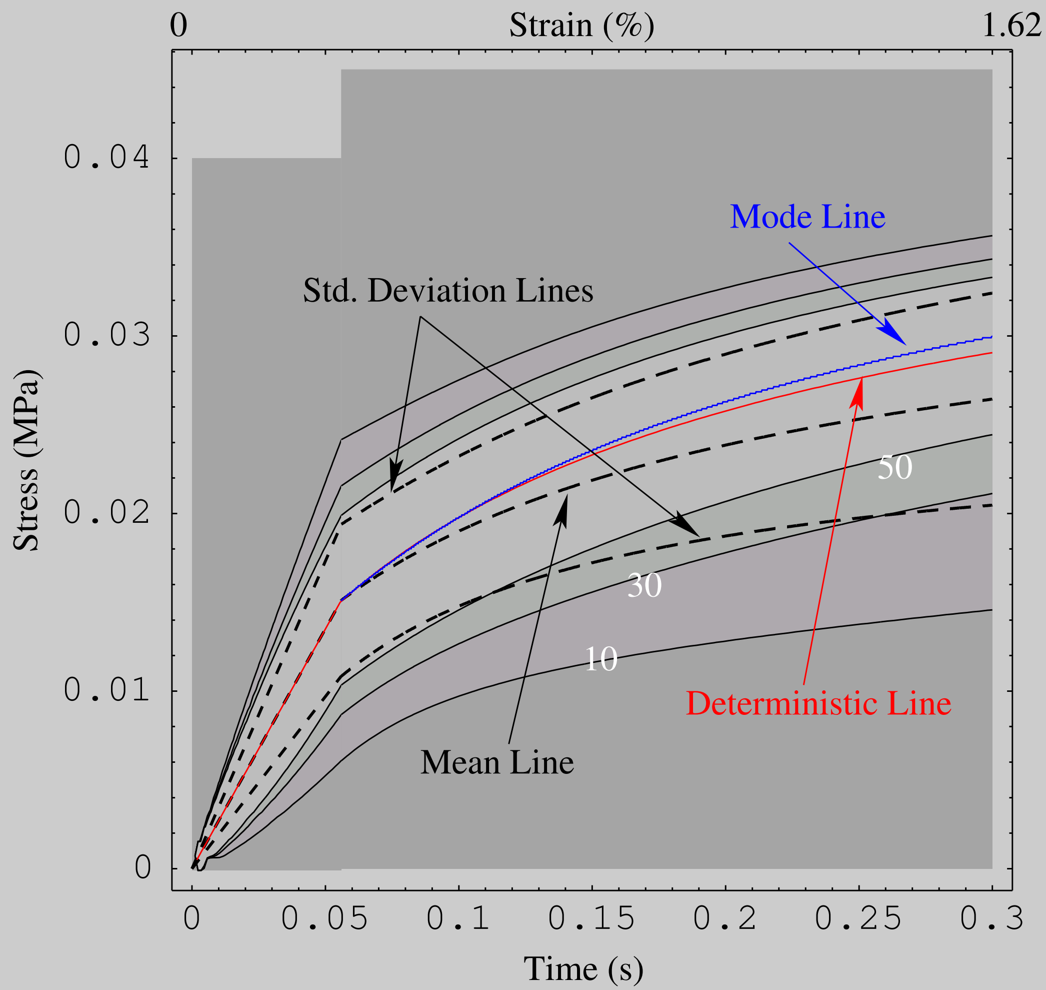
Low OCR Cam Clay with Random G



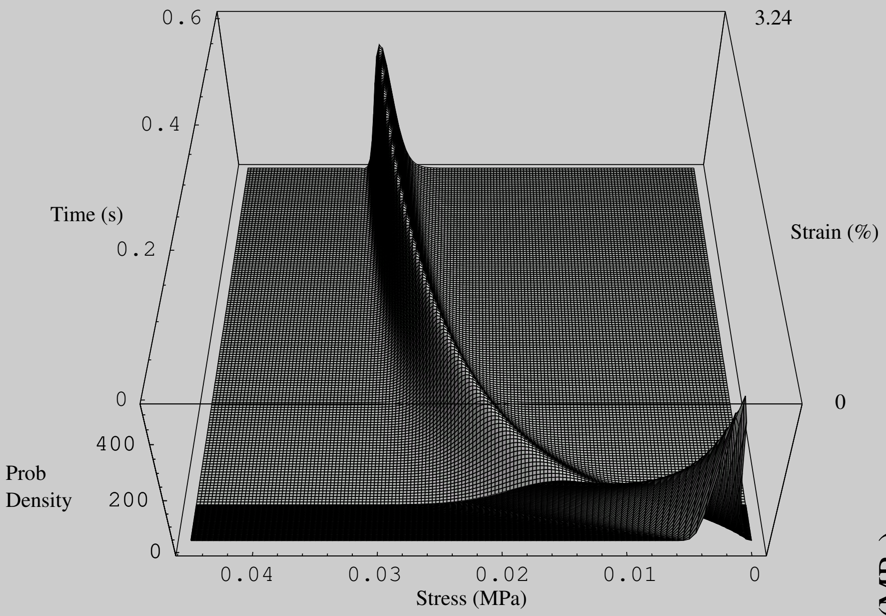


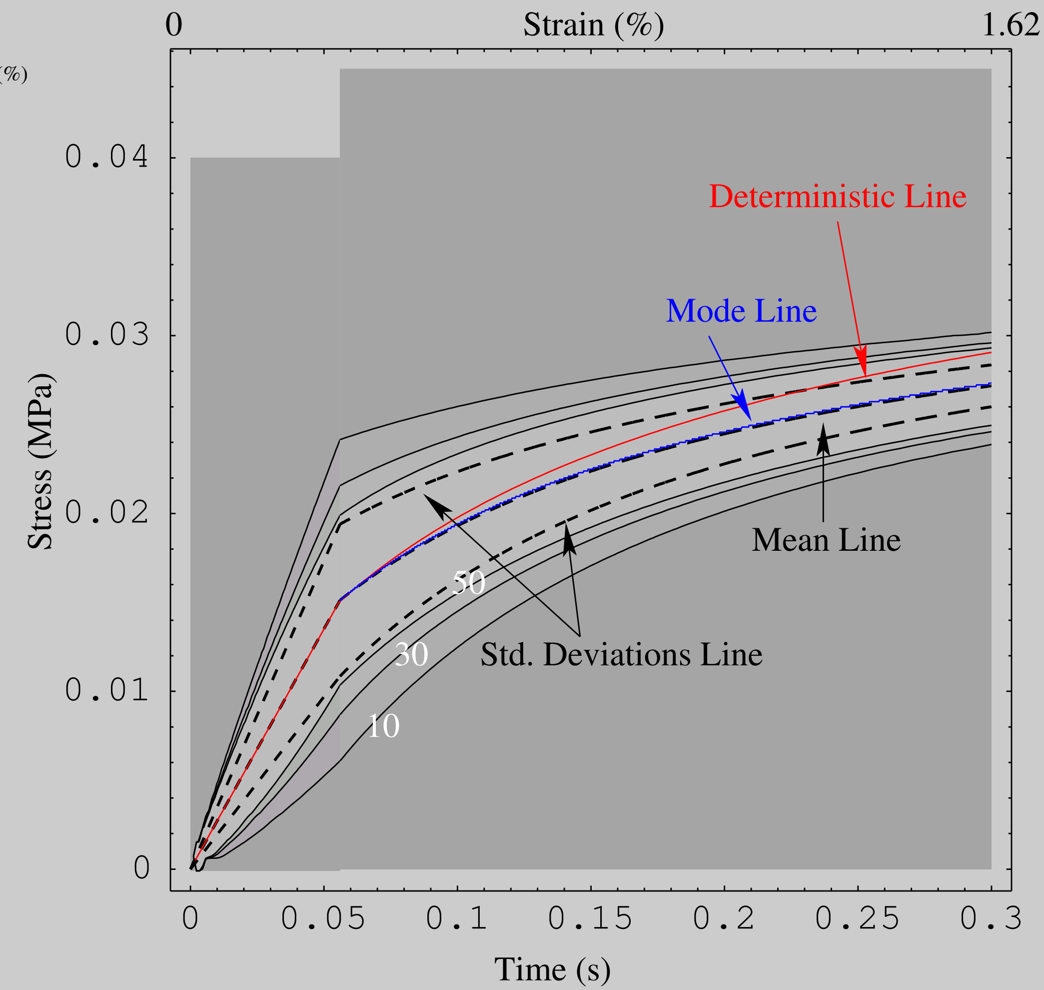
Low OCR Cam Clay Response with Random G and Random M



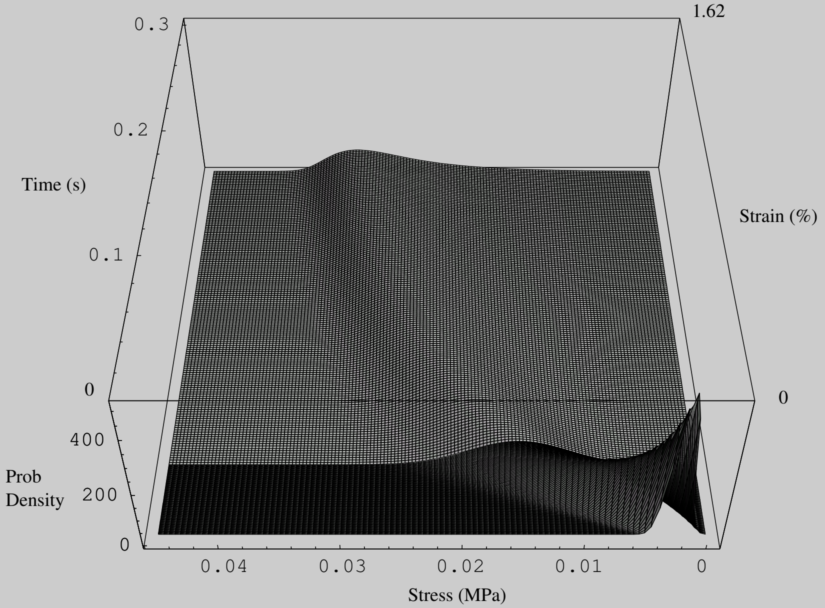


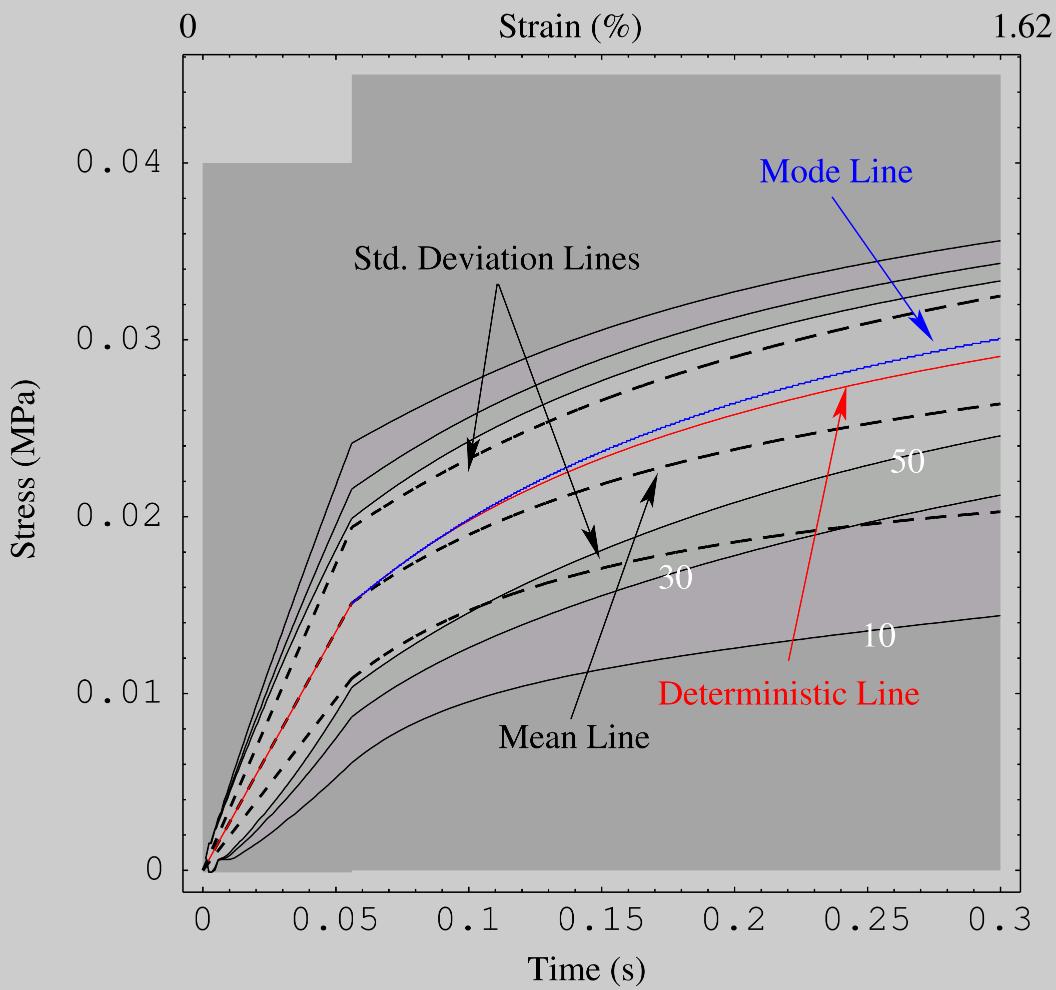
Low OCR Cam Clay Response with Random G and Random p_0



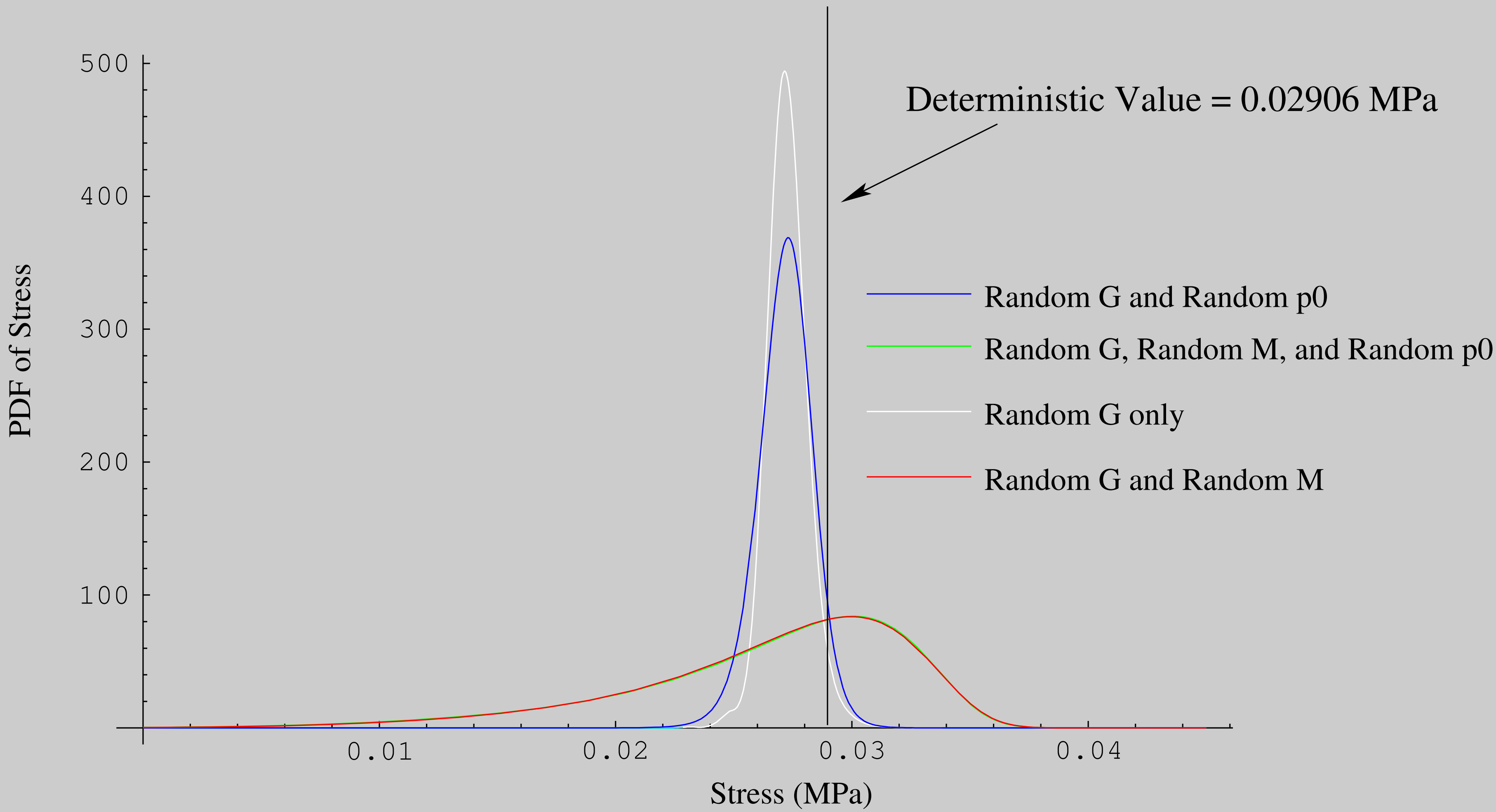


Low OCR Cam Clay Response with Random G, Random M and Random p_0

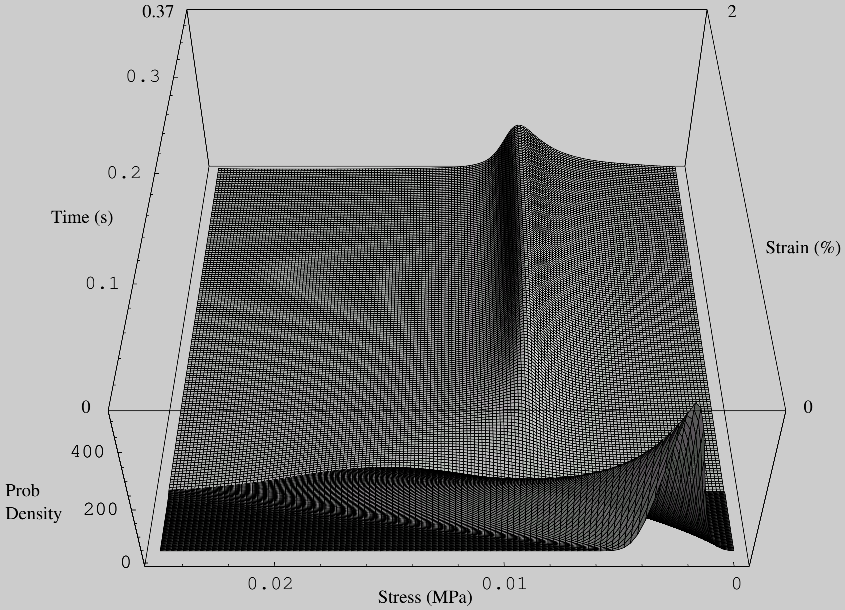


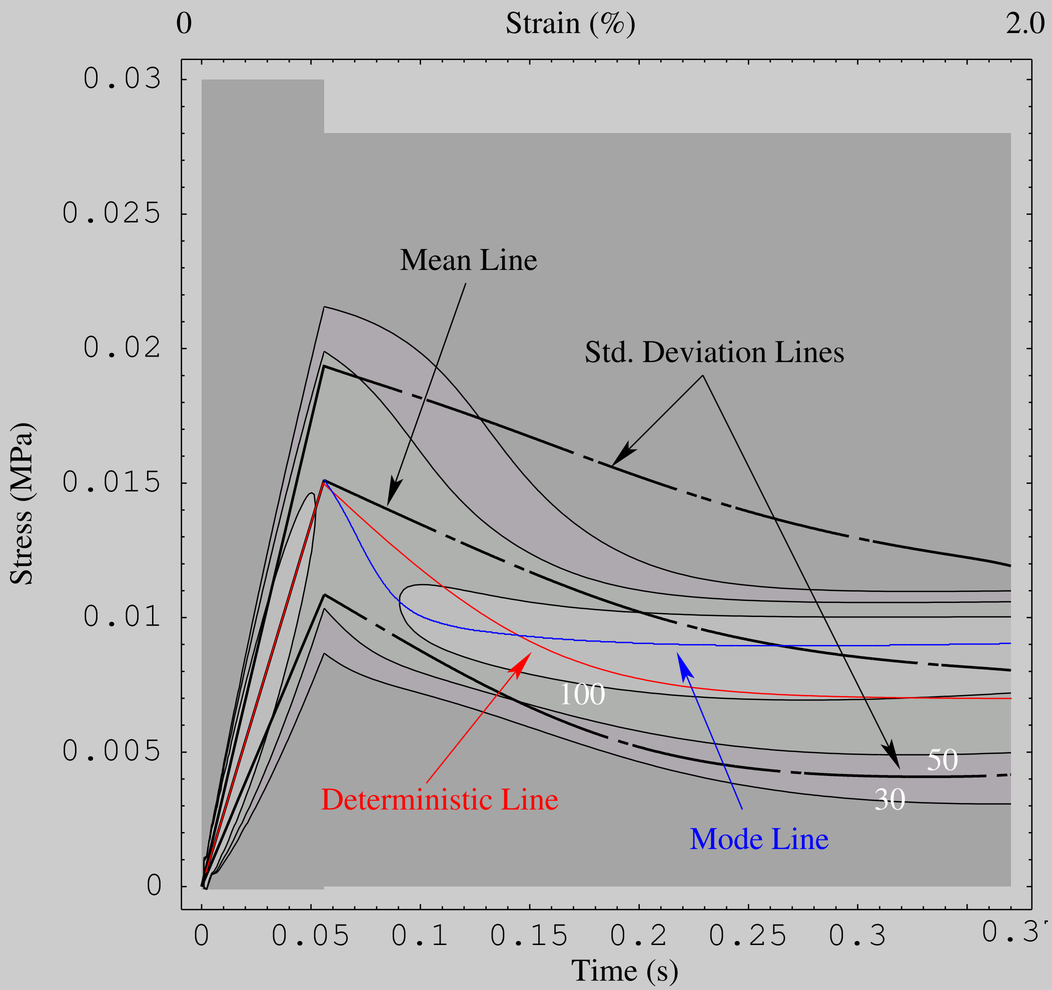


Low OCR Cam Clay Predictions at $\epsilon = 1.62$ %



High OCR Cam Clay Response with Random G and Random M





Summary

- Expression for evolution of probability densities of stress was derived for any general 1-D elastic-plastic constitutive rate equation.

- This method doesn't require repetitive use of computationally expensive deterministic elastic-plastic model and doesn't suffer from 'closure problem' associated with regular perturbation approach.

- Furthermore, the developed expression is linear and derterministic PDE whereas the constitutive rate equation is random and nonlinear.

- Current work is going on in extending this method to 3-D and incorporating it to the formulation of stochastic elastic-plastic finite element method.