On Uncertainty of Elastic-Plastic Simulations

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Talk Overview

• Motivation: Uncertainty in (geo–) material modeling and simulations

• Previous work

• Proposed formulation and solution (Forward Kolmogorov or Fokker-Planck equation)

• Select results and verifications
  – Elastic
  – Drucker-Prager Linear Hardening
  – Cam Clay
Motivation

- Material behavior is stochastic, both spatially and point-wise,

- How is failure mechanics of solids and structures affected by that stochasticity?

- Can the Stochastic approach to Elasto–Plasticity offer more information about the failure of a *particular* solid

- Can the Stochastic approach to Elasto–Plasticity offer more information (missing link) about the failure of *general* solids (and structures)
Motivation: Typical Soil Profile
### Motivation: Point-wise Variation

<table>
<thead>
<tr>
<th>(a) Soil property</th>
<th>Soil type</th>
<th>pdf</th>
<th>Mean</th>
<th>COV(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cone resistance</td>
<td>Sand Clay</td>
<td>LN</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Clay</td>
<td>N/LN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Undrained shear strength</td>
<td>Clay (triaxial)</td>
<td>LN</td>
<td>*</td>
<td>5-20</td>
</tr>
<tr>
<td></td>
<td>Clay (index Su)</td>
<td>LN</td>
<td></td>
<td>10-35</td>
</tr>
<tr>
<td></td>
<td>Clayey silt</td>
<td>N</td>
<td></td>
<td>10-30</td>
</tr>
<tr>
<td>Ratio $S_u/\sigma_{\sigma'}$</td>
<td>Clay</td>
<td>N/LN</td>
<td>*</td>
<td>5-15</td>
</tr>
<tr>
<td>Plastic limit</td>
<td>Clay</td>
<td>N</td>
<td>0.13-0.23</td>
<td>3-20</td>
</tr>
<tr>
<td>Liquid limit</td>
<td>Clay</td>
<td>N</td>
<td>0.30-0.80</td>
<td>3-20</td>
</tr>
<tr>
<td>Submerged unit weight</td>
<td>All soils</td>
<td>N</td>
<td>5-11 (kN/m3)</td>
<td>0-10</td>
</tr>
<tr>
<td>Friction angle</td>
<td>Sand</td>
<td>N</td>
<td>*</td>
<td>2-5</td>
</tr>
<tr>
<td>Void ratio, porosity, initial void ratio</td>
<td>All soils</td>
<td>N</td>
<td>*</td>
<td>7-30</td>
</tr>
<tr>
<td>Over consolidation ratio</td>
<td>Clay</td>
<td>N/LN</td>
<td>*</td>
<td>10-35</td>
</tr>
</tbody>
</table>
Uncertainty in Geomechanics

- Uncertainty of geomaterial properties:
  a Natural variability of soil deposits
  b Sampling error
  c Testing error

- Aleatory uncertainty $\rightarrow$ inherent variation associated with the physical system of the environment (variation in external excitation, material properties...). Also known as irreducible uncertainty, variability and stochastic uncertainty. (a)

- Epistemic uncertainty $\rightarrow$ potential deficiency in any phase of the modeling process that is due to lack of knowledge (poor understanding of mechanics...). Also known as reducible uncertainty, model form uncertainty and subjective uncertainty. (b, c)
Previous Work

• Linear algebraic relations (linear elastic) $\rightarrow$ analytical expressions:
  – variable transformation (Montgomery and Runger 2003)
  – cumulant expansion method (Gardiner 2004)

• Nonlinear differential equations $\rightarrow$
Objectives of the Proposed Method

- Overcome the disadvantages of the perturbation and Monte Carlo approaches,

- Capable of carrying out sensitivity analysis at a point–location scale, when material parameter are modeled as random variables,

- Obtain probabilistic behavior of spatial average form (upscaled form) of constitutive rate equation when material properties are modeled as random field.
Problem Statement: 3D

- The general 3-D constitutive rate equation - a nonlinear ODE system with random coefficient and random forcing

\[
\frac{d\sigma_{ij}(t)}{dt} = D_{ijkl}\frac{d\epsilon_{kl}(t)}{dt}
\]

\[
D_{ijkl} = \begin{cases} 
D_{ijkl}^e & \text{when } f < 0 \lor (f = 0 \land df < 0) \\
D_{ijkl}^e - \frac{D_{ijmn}^e}{\partial\sigma_{rs}} \frac{\partial U}{\partial\sigma_{tu}} \frac{\partial f}{\partial\sigma_{pq}} D_{pqkl}^e & \text{when } f = 0 \lor df = 0
\end{cases}
\]
Problem Statement: 1D

• 1-D – a nonlinear ODE, random coefficient and random forcing

\[
\frac{d\sigma(t)}{dt} = \beta(\sigma, D, q, r; t) \frac{d\epsilon(t)}{dt} = \eta(\sigma, D, q, r, \epsilon; t)
\]

with an initial condition \( \sigma(0) = \sigma_0 \)
Stochastic Continuity Equation

- The 1-D constitutive equation visualization: from each initial point in $\sigma$-space a trajectory starts out which describes the corresponding solution of the stochastic process

- Consider a cloud of initial points (described by density $\rho(\sigma, 0)$ in $\sigma$-space): movement of all these points is dictated by the constitutive equation, the phase density $\rho$ varies in time according to a continuity equation (Liouville equation):

$$
\frac{\partial \rho(\sigma(t), t)}{\partial t} = -\frac{\partial}{\partial \sigma} \eta[\sigma(t), D, q, r, \epsilon(t)].\rho[\sigma(t), t]
$$

with initial condition

$$
\rho(\sigma, 0) = \delta(\sigma - \sigma_0)
$$
Fokker-Planck Equation

Writing the continuity equation in ensemble average form and using Van Kampen’s Lemma \( \langle \rho(h, t) \rangle \equiv P(h, t) \) yields the following Fokker-Planck equation:

\[
\frac{\partial P(\sigma(t), t)}{\partial t} = - \frac{\partial}{\partial \sigma} \left[ \left\{ \left\langle \eta(\sigma(t), D, q, r, \epsilon(t)) \right\rangle \right\} \right] \\
+ \int_0^t d\tau \text{Cov}_0 \left[ \frac{\partial \eta(\sigma(t), D, q, r, \epsilon(t))}{\partial \sigma} \right] \eta(\sigma(t - \tau), D, q, r, \epsilon(t - \tau)) \right\} P(\sigma(t), t) \\
+ \frac{\partial^2}{\partial \sigma^2} \left[ \left\{ \int_0^t d\tau \text{Cov}_0 \left[ \eta(\sigma(t), D, q, r, \epsilon(t)); \eta(\sigma(t - \tau), D, q, r, \epsilon(t - \tau)) \right] \right\} P(\sigma(t), t) \right] 
\]
Solution of Fokker-Planck Equation

• The Fokker-Planck equation → advection-diffusion equation:

\[
\frac{\partial P(\sigma, t)}{\partial t} = -\frac{\partial}{\partial \sigma} \left[ N(1)P(\sigma, t) - \frac{\partial}{\partial \sigma} \{ N(2)P(\sigma, t) \} \right] = -\frac{\partial \zeta}{\partial \sigma}
\]

• Initial condition – deterministic (Dirac delta function) or random

\[ P(\sigma, 0) = \delta(\sigma) \]

• Boundary condition – reflecting (conserve probability mass or no probability current flow)

\[ \zeta(\sigma, t) \big|_{AtBoundaries} = 0 \]

• The Fokker-Planck equation solution → Finite Difference Technique
Numerical Scheme

- The Fokker-Planck equation was solved using *Method of Lines* by semi-discretizing the stress domain using *Finite Difference Technique*.
At Intermediate Node $i$

$$\frac{\partial P^{(i)}}{\partial t} = + P^{(i-1)} \left[ \frac{N^{(1)}_{(1)}}{2\Delta\sigma} + \frac{N^{(i)}_{(2)}}{\Delta\sigma^2} - \frac{1}{\Delta\sigma} \frac{\partial N^{(i)}_{(2)}}{\partial\sigma} \right]$$

$$- P^{(i)} \left[ \frac{\partial N^{(i)}_{(1)}}{\partial\sigma} + 2\frac{N^{(i)}_{(2)}}{\Delta\sigma^2} - \frac{\partial^2 N^{(i)}_{(2)}}{\partial\sigma^2} \right]$$

$$+ P^{(i+1)} \left[ -\frac{N^{(i)}_{(1)}}{2\Delta\sigma} + \frac{N^{(i)}_{(2)}}{\Delta\sigma^2} + \frac{1}{\Delta\sigma} \frac{\partial N^{(i)}_{(2)}}{\partial\sigma} \right]$$

- Not a very efficient scheme

- Possible improvement through adaptivity

- Also considering Reduced Order Modeling (ROM)
Numerical Scheme

• Introducing the BC at the left boundary

\[ P^{(1)} = P^{(2)} \]

\[ \begin{bmatrix}
\frac{N^{(1)}_2}{\Delta \sigma} \\
\frac{N^{(1)}_2}{\Delta \sigma} - \frac{\partial N^{(1)}_2}{\partial \sigma}
\end{bmatrix}
\]

• and at the right boundary

\[ P^{(n)} = P^{(n-1)} \]

\[ \begin{bmatrix}
\frac{N^{(n)}_2}{\Delta \sigma} \\
-\frac{N^{(n)}_2}{\Delta \sigma} - \frac{\partial N^{(n)}_2}{\partial \sigma}
\end{bmatrix}
\]

• The semi-discretized PDE (i.e. a set of simultaneous ODEs) was solved using

ODE solver available in Mathematica
Elastic Response with Random $G$

- General form of elastic constitutive rate equation
  \[ \frac{d\sigma_{12}}{dt} = 2G\frac{d\epsilon_{12}}{dt} = \eta(G, \epsilon_{12}; t) \]

- The advection and diffusion coefficients of FPE are
  \[ N_{(1)} = 2\frac{d\epsilon_{12}}{dt} < G > \quad ; \quad N_{(2)} = 4t \left( \frac{d\epsilon_{12}}{dt} \right)^2 Var[G] \]
Verification of Elastic Response

Variable Transformation

![Graph showing verification of elastic response with variable transformation and probability density analysis.](image-url)
Drucker-Prager Associative Linear Hardening with Random $G$

- The general form of Drucker-Prager elastic-plastic associative linear hardening constitutive rate equation

\[
\frac{d\sigma_{12}}{dt} = G^{ep} \frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)
\]

- The advection and diffusion coefficients of FPE are

\[
N^{(1)} = \frac{d\epsilon_{12}}{dt} \left( 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'} \right)
\]

\[
N^{(2)} = t \left( \frac{d\epsilon_{12}}{dt} \right)^2 \text{Var} \left[ 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'} \right]
\]
Drucker-Prager Associative Linear Hardening with Random $G$
Comparing Uncertainty in Elastic and Elastic–Plastic Response
Cam Clay Constitutive Model

- The general form of Cam Clay 1-D shear constitutive rate equation

\[
\frac{d\sigma_{12}}{dt} = G^{ep} \frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)
\]

where \( \eta \) has the form:

\[
\eta = 2G - \left[ 36 \frac{G^2}{M^4} \sigma_{12}^2 \right] \frac{(1 + e_0)p(2p - p_0)^2}{\kappa} + \left( 18 \frac{G}{M^4} \right) \sigma_{12}^2 + \frac{1 + e_0}{\lambda - \kappa} pp_0(2p - p_0)
\]

- The advection and diffusion coefficients of FPE are

\[
N^{(i)}_{(1)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau \text{cov} \left[ \frac{\partial \eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t - \tau) \right]
\]

\[
N^{(i)}_{(2)} = \int_0^t d\tau \text{cov} \left[ \eta^{(i)}(t); \eta^{(i)}(t - \tau) \right]
\]
Low OCR Cam Clay with Random $G$
Low OCR Cam Clay Response with Random $G$ and Random $M$
Low OCR Cam Clay Response with Random $G$ and Random $p_0$

![Graph showing stress-strain relationship with probabilistic density function and mean line.]
Low OCR Cam Clay Response with Random $G$, Random $M$ and Random $p_0$
Low OCR Cam Clay Predictions at $\epsilon = 1.62 \%$

Deterministic Value = 0.02906 MPa

- Random G and Random p0
- Random G, Random M, and Random p0
- Random G only
- Random G and Random M

Stress (MPa)

PDF of Stress
High OCR Cam Clay Response with Random $G$ and Random $M$

\[\begin{array}{c}
\text{Time (s)} \\
\text{Stress (MPa)} \\
\text{Strain (%)} \\
\text{Prob Density}
\end{array}\]

\[\begin{array}{c}
0.005 \\
0.01 \\
0.015 \\
0.02 \\
0.025 \\
0.03 \\
0.035 \\
0.04 \\
0.045
\end{array}\]

\[\begin{array}{c}
0 \\
100 \\
300 \\
500 \\
700 \\
900 \\
1100 \\
1300 \\
1500
\end{array}\]

\[\begin{array}{c}
0 \\
100 \\
300 \\
50 \\
30
\end{array}\]
Summary

• Expression for evolution of probability densities of stress was derived for any general 1-D elastic-plastic constitutive rate equation.

• This method doesn’t require repetitive use of computationally expensive deterministic elastic-plastic model and doesn’t suffer from ’closure problem’ associated with regular perturbation approach.

• Furthermore, the developed expression is linear and deterministic PDE whereas the constitutive rate equation is random and non-linear.

• Current work is going on in extending this method to 3-D and incorporating it to the formulation of stochastic elastic-plastic finite element method.