On Uncertainty of Elastic-Plastic Simulations

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Talk Overview

- Motivation: Uncertainty in (geo–) material modeling and simulations
- Previous work
- Proposed formulation and solution (Forward Kolmogorov or Fokker-Planck equation)
- Select results and verifications
  - Elastic
  - Drucker-Prager Linear Hardening
  - Cam Clay

Motivation

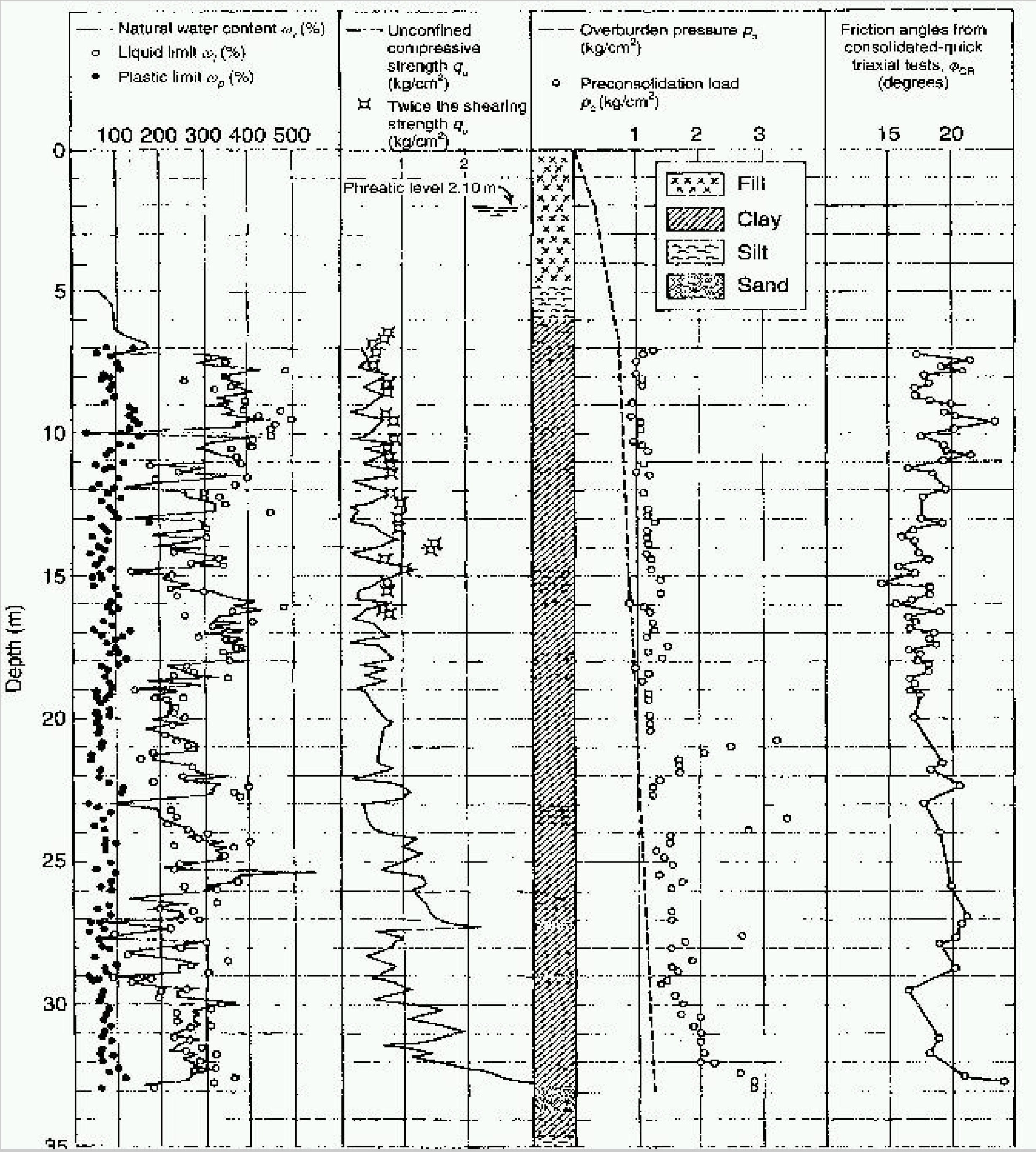
- Material behavior is stochastic, both spatially and point-wise,

- How is failure mechanics of solids and structures affected by that stochasticity?

- Can the Stochastic approach to Elasto–Plasticity offer more information about the failure of a *particular* solid

- Can the Stochastic approach to Elasto–Plasticity offer more information (missing link) about the failure of *general* solids (and structures)

Motivation: Typical Soil Profile



Motivation: Point-wise Variation

(a) Soil property	Soil type	pdf	Mean	*COV*(%)
Cone resistance	Sand Clay	LN	*	*
	Clay	N/LN		
Undrained shear strength	Clay (triaxial)	LN	*	5–20
	Clay (index Su)	LN		10–35
	Clayey silt	N		10–30
Ratio S_u/σ'_{vo}	Clay	N/LN	*	5- 15
Plastic limit	Clay	N	0.13–0.23	3–20
Liquid limit	Clay	N	0.30--0.80	3–20
Submerged unit weight	All soils	N	5–11 (kN/m3)	0–10
Friction angle	Sand	N	*	2–5
Void ratio, porosity, initial void ratio	All soils	N	*	7–30
Over consolidation ratio	Clay	N/LN	*	10--35

Uncertainty in Geomechanics

- Uncertainty of geomaterial properties:

  a Natural variability of soil deposits
  b Sampling error
  c Testing error

- Aleatory uncertainty → inherent variation associated with the physical system of the environment (variation in external excitation, material properties...). Also know known as irreducible uncertainty, variability and stochastic uncertainty. (**a**)

- Epistemic uncertainty → potential deficiency in any phase of the modeling process that is due to lack of knowledge (poor understanding of mechanics...). Also known as reducible uncertainty, model form uncertainty and subjective uncertainty. (**b, c**)

Previous Work

- Linear algebraic relations (linear elastic) $\rightarrow$ analytical expressions:
  - variable transformation (Montgomery and Runger 2003)
  - cumulant expansion method (Gardiner 2004)

- Nonlinear differential equations $\rightarrow$
  - Monte Carlo analysis (Schueller 1997, De Lima et al, 2001, Mellah et al. 2000...)
  - Perturbation approach (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 2997, Mellah et al, 2000)

Objectives of the Proposed Method

- Overcome the disadvantages of the perturbation and Monte Carlo approaches,

- Capable of carrying out sensitivity analysis at a point–location scale, when material parameter are modeled as random variables,

- Obtain probabilistic behavior of spatial average form (upscaled form) of constitutive rate equation when material properties are modeled as random field.

Problem Statement: 3D

- The general 3-D constitutive rate equation - a nonlinear ODE system with random coefficient and random forcing

$$\frac{d\sigma_{ij}(t)}{dt} = D_{ijkl} \frac{d\epsilon_{kl}(t)}{dt}$$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{when } f < 0 \vee (f = 0 \wedge df < 0) \\ \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn} \dfrac{\partial U}{\partial \sigma_{mn}} \dfrac{\partial f}{\partial \sigma_{pq}} D^{el}_{pqkl}}{\dfrac{\partial f}{\partial \sigma_{rs}} D^{el}_{rstu} \dfrac{\partial U}{\partial \sigma_{tu}} - \dfrac{\partial f}{\partial q_*} r_*} & \text{when } f = 0 \vee df = 0 \end{cases}$$

Problem Statement: 1D

- 1-D – a nonlinear ODE, random coefficient and random forcing

$$\frac{d\sigma(t)}{dt} = \beta(\sigma, D, q, r; t)\frac{d\epsilon(t)}{dt} = \eta(\sigma, D, q, r, \epsilon; t)$$

with an initial condition $\sigma(0) = \sigma_0$

Stochastic Continuity Equation

- The 1-D constitutive equation visualization: from each initial point in σ-space a trajectory starts out which describes the corresponding solution of the stochastic process

- Consider a cloud of initial points (described by density $\rho(\sigma, 0)$ in σ-space): movement of all these points is dictated by the constitutive equation, the phase density ρ varies in time according to a continuity equation (Liouville equation):

$$\frac{\partial \rho(\sigma(t), t)}{\partial t} = -\frac{\partial}{\partial \sigma} \eta[\sigma(t), D, q, r, \epsilon(t)].\rho[\sigma(t), t]$$

with initial condition

$$\rho(\sigma, 0) = \delta(\sigma - \sigma_0)$$

Fokker-Planck Equation

- Writing the continuity equation in ensemble average form and using Van Kampen's Lemma ($< \rho(h,t) >= P(h,t)$) yields the following Fokker-Planck equation:

$$\begin{aligned}
\frac{\partial P(\sigma(t),t)}{\partial t} = & - \frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(t),D,q,r,\epsilon(t))\right\rangle\right.\right. \\
& + \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(t),D,q,r,\epsilon(t))}{\partial \sigma};\right. \\
& \left.\left.\left.\eta(\sigma(t-\tau),D,q,r,\epsilon(t-\tau)\right]\right\} P(\sigma(t),t)\right] \\
& + \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(t),D,q,r,\epsilon(t));\right.\right.\right. \\
& \left.\left.\left.\eta(\sigma(t-\tau),D,q,r,\epsilon(t-\tau))\right]\right\} P(\sigma(t),t)\right]
\end{aligned}$$

Solution of Fokker-Planck Equation

- The Fokker-Planck equation $\rightarrow$ advection-diffusion equation:

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition – deterministic (Dirac delta function) or random

$$P(\sigma,0) = \delta(\sigma)$$

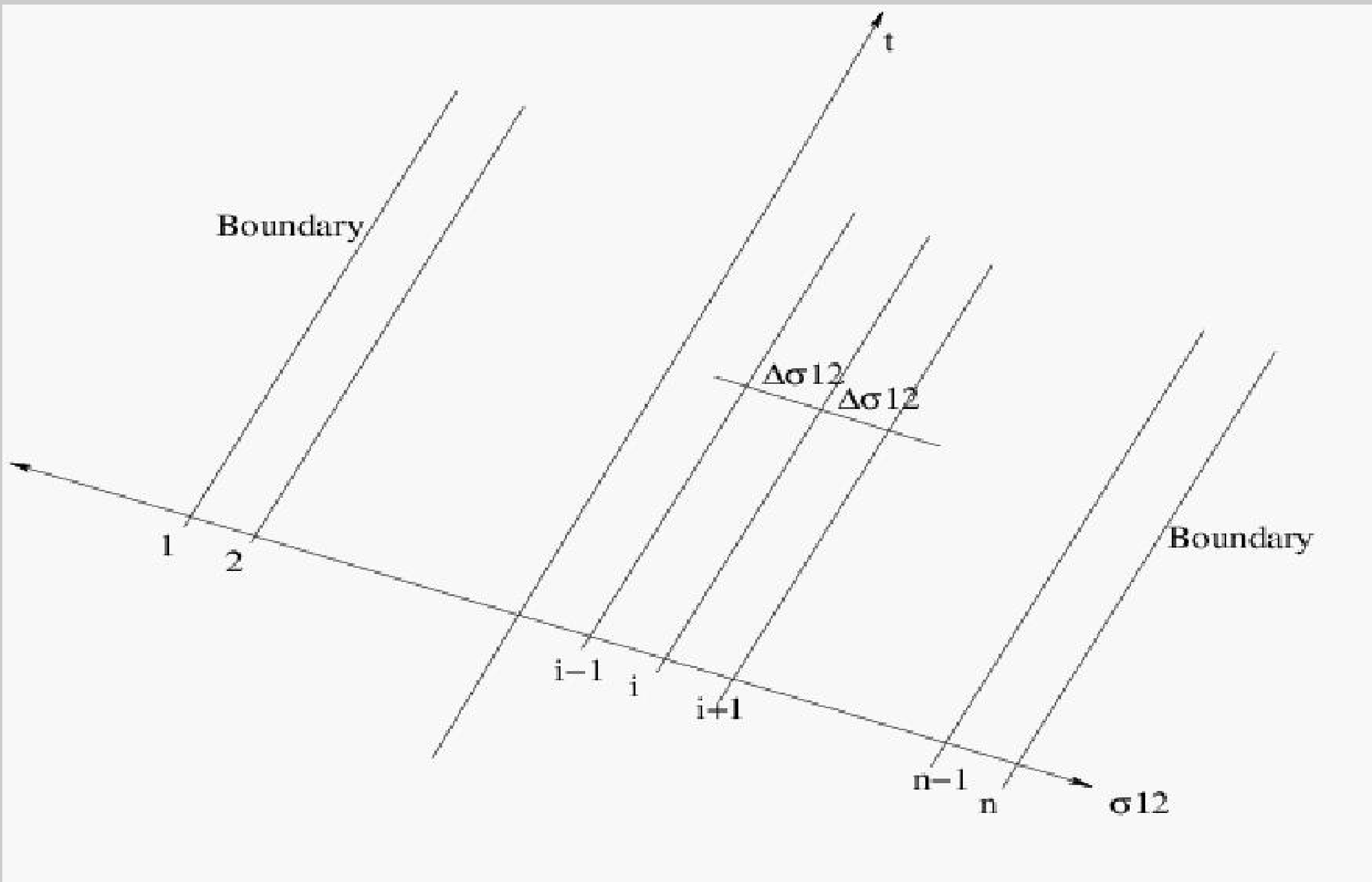
- Boundary condition – reflecting (conserve probability mass or no probability current flow)

$$\zeta(\sigma,t)|_{AtBoundaries} = 0$$

- The Fokker-Planck equation solution $\rightarrow$ *Finite Difference Technique*

Numerical Scheme

- The Fokker-Planck equation was solved using *Method of Lines* by semi-discretizing the stress domain using *Finite Difference Technique*



At Intermediate Node i

$$
\begin{aligned}
\frac{\partial P^{(i)}}{\partial t} = \quad & + \quad P^{(i-1)} \left[\frac{N^{(i)}_{(1)}}{2\Delta\sigma} + \frac{N^{(i)}_{(2)}}{\Delta\sigma^2} - \frac{1}{\Delta\sigma}\frac{\partial N^{(i)}_{(2)}}{\partial\sigma} \right] \\
& - \quad P^{(i)} \left[\frac{\partial N^{(i)}_{(1)}}{\partial\sigma} + 2\frac{N^{(i)}_{(2)}}{\Delta\sigma^2} - \frac{\partial^2 N^{(i)}_{(2)}}{\partial\sigma^2} \right] \\
& + \quad P^{(i+1)} \left[-\frac{N^{(i)}_{(1)}}{2\Delta\sigma} + \frac{N^{(i)}_{(2)}}{\Delta\sigma^2} + \frac{1}{\Delta\sigma}\frac{\partial N^{(i)}_{(2)}}{\partial\sigma} \right]
\end{aligned}
$$

- Not a very efficient scheme
- Possible improvement through adaptivity
- Also considering Reduced Order Modeling (ROM)

Numerical Scheme

- Introducing the BC at the left boundary

$$P^{(1)} = P^{(2)} \left[\frac{\dfrac{N^{(1)}_{(2)}}{\Delta\sigma}}{N^{(1)}_{(1)} + \dfrac{N^{(1)}_{(2)}}{\Delta\sigma} - \dfrac{\partial N^{(1)}_{(2)}}{\partial\sigma}} \right]$$

- and at the right boundary

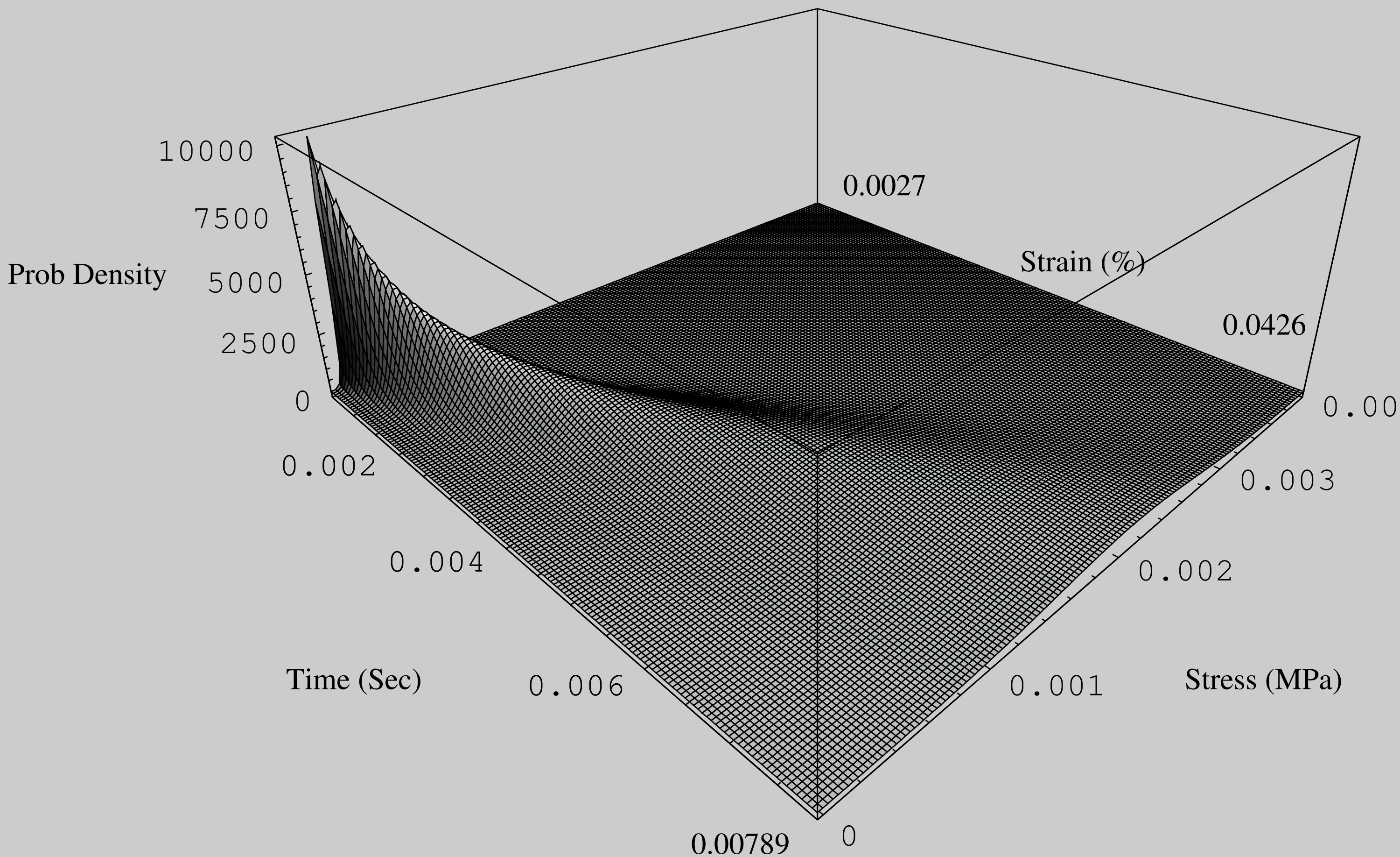
$$P^{(n)} = P^{(n-1)} \left[\frac{\dfrac{N^{(n)}_{(2)}}{\Delta\sigma}}{-N^{(n)}_{(1)} + \dfrac{N^{(n)}_{(2)}}{\Delta\sigma} - \dfrac{\partial N^{(n)}_{(2)}}{\partial\sigma}} \right]$$

- The semi-discretized PDE (i.e. a set of simultaneous ODEs) was solved using ODE solver available in *Mathematica*

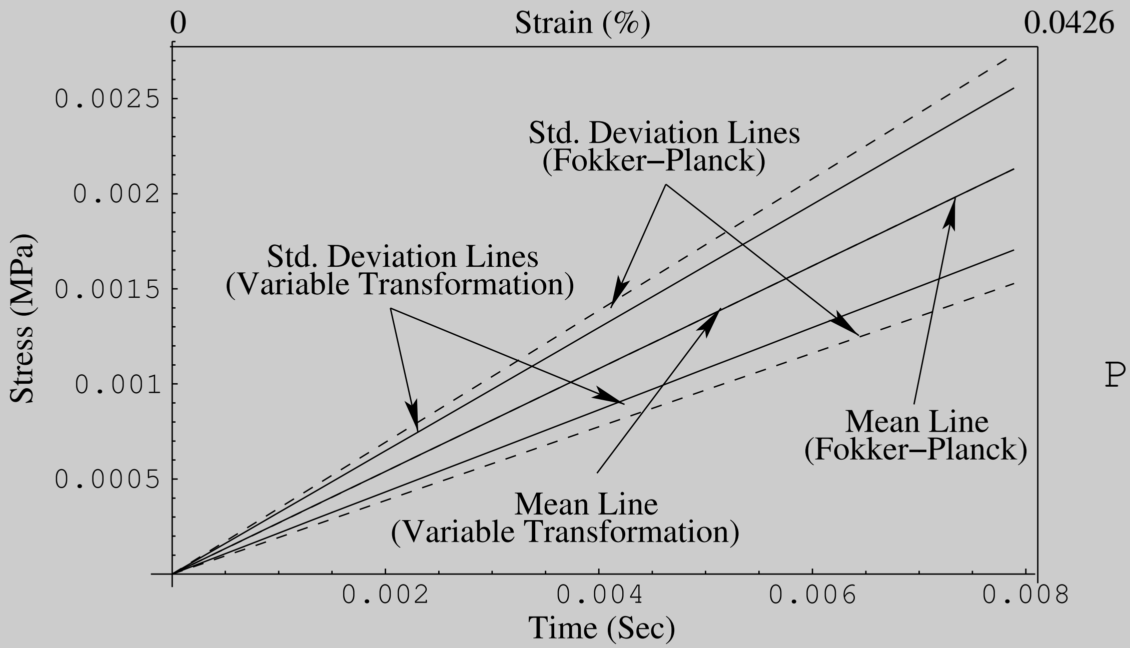
Elastic Response with Random G

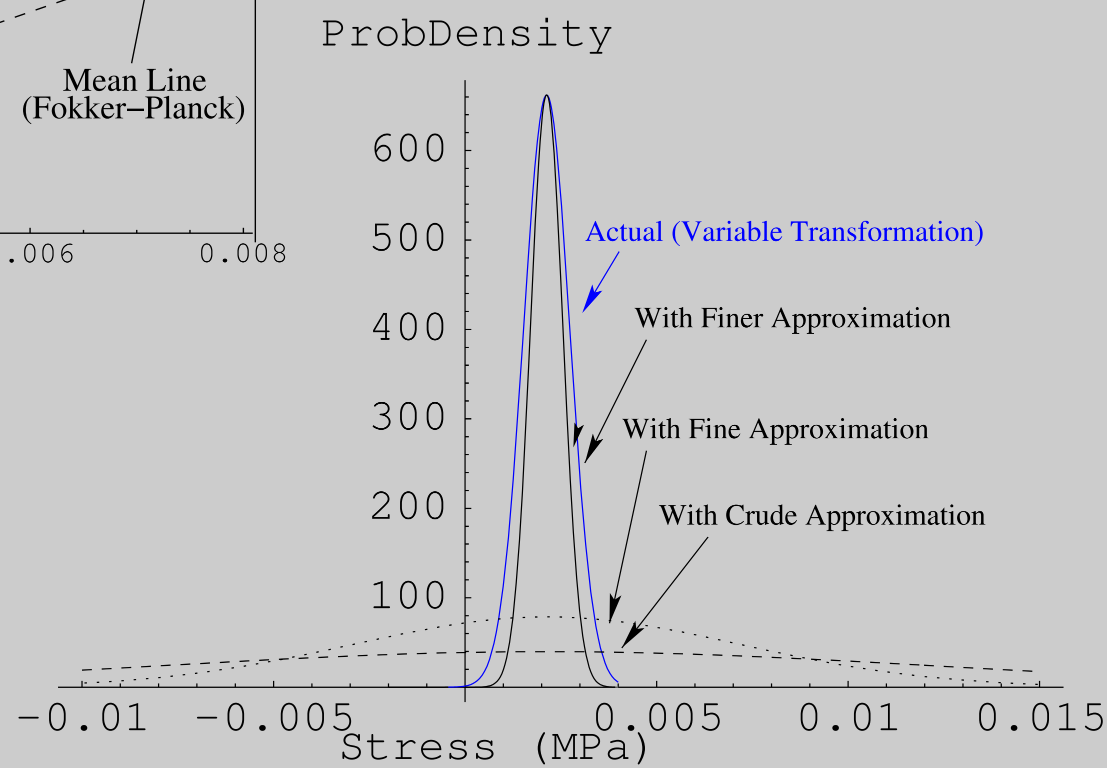
- General form of elastic constitutive rate equation
  $d\sigma_{12}/dt = 2G d\epsilon_{12}/dt = \eta(G, \epsilon_{12}; t)$

- The advection and diffusion coefficients of FPE are
  $N_{(1)} = 2d\epsilon_{12}/dt < G >$; $N_{(2)} = 4t\left(d\epsilon_{12}/dt\right)^2 Var[G]$



Verification of Elastic Response
Variable Transformation





Drucker-Prager Associative Linear Hardening with Random G

- The general form of Drucker-Prager elastic-plastic associative linear hardening constitutive rate equation

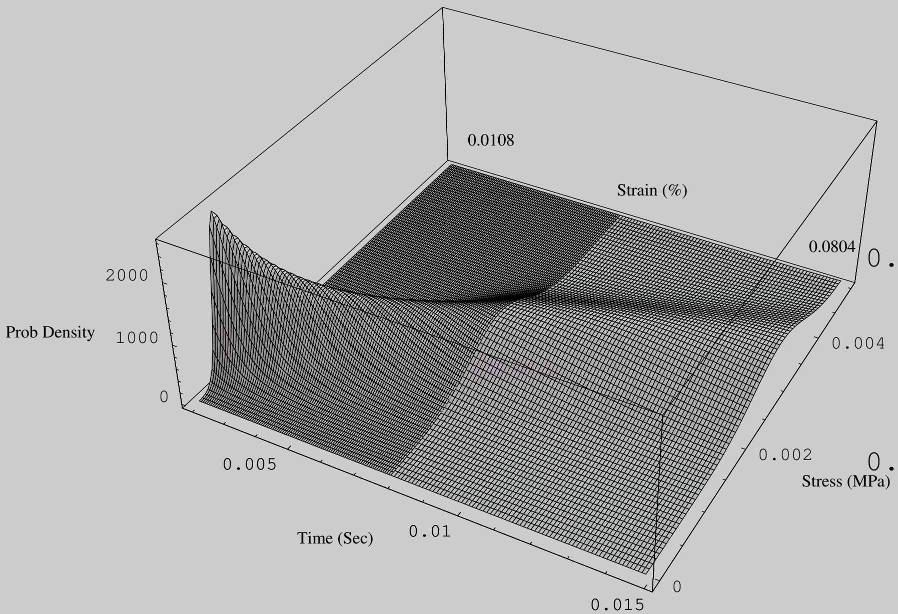
$$\begin{aligned}\frac{d\sigma_{12}}{dt} &= G^{ep}\frac{d\epsilon_{12}}{dt} \\ &= \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)\end{aligned}$$

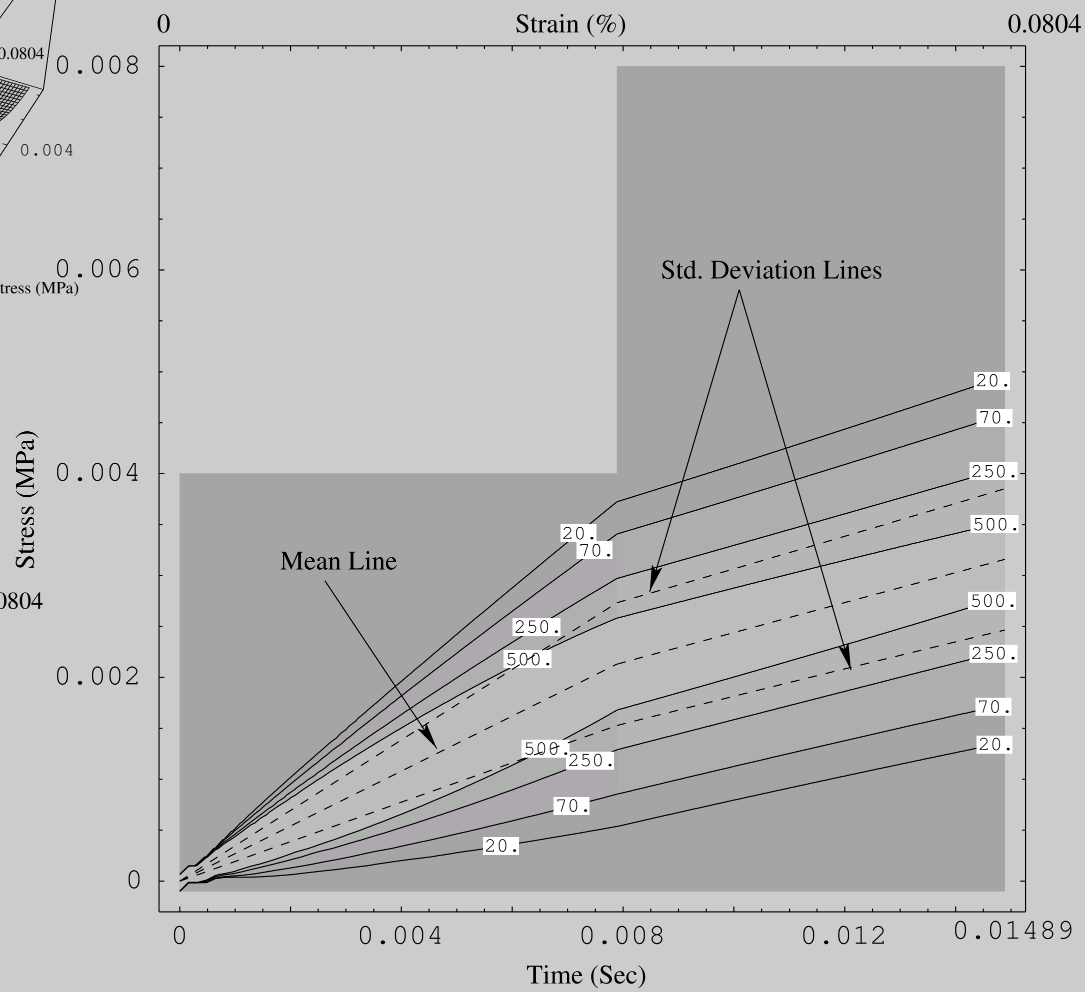
- The advection and diffusion coefficients of FPE are

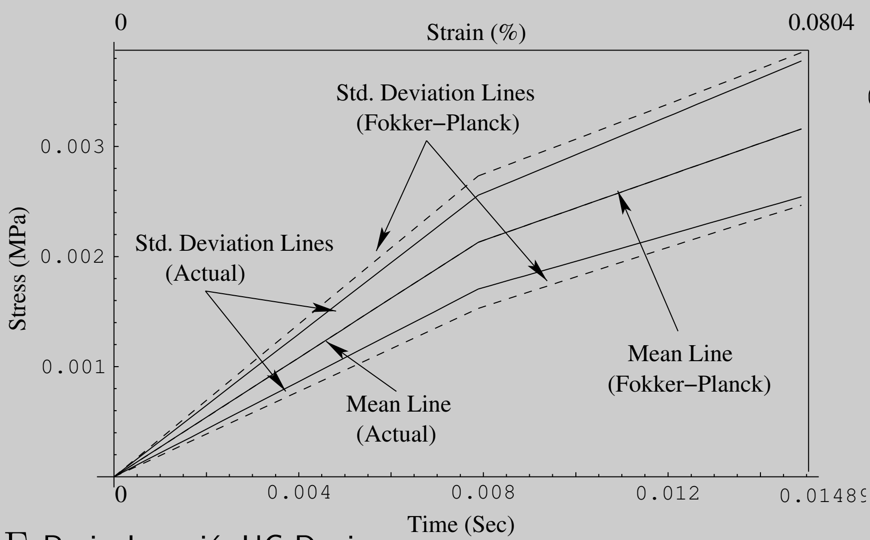
$$N_{(1)} = \frac{d\epsilon_{12}}{dt}\left\langle 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right\rangle$$

$$N_{(2)} = t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var\left[2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right]$$

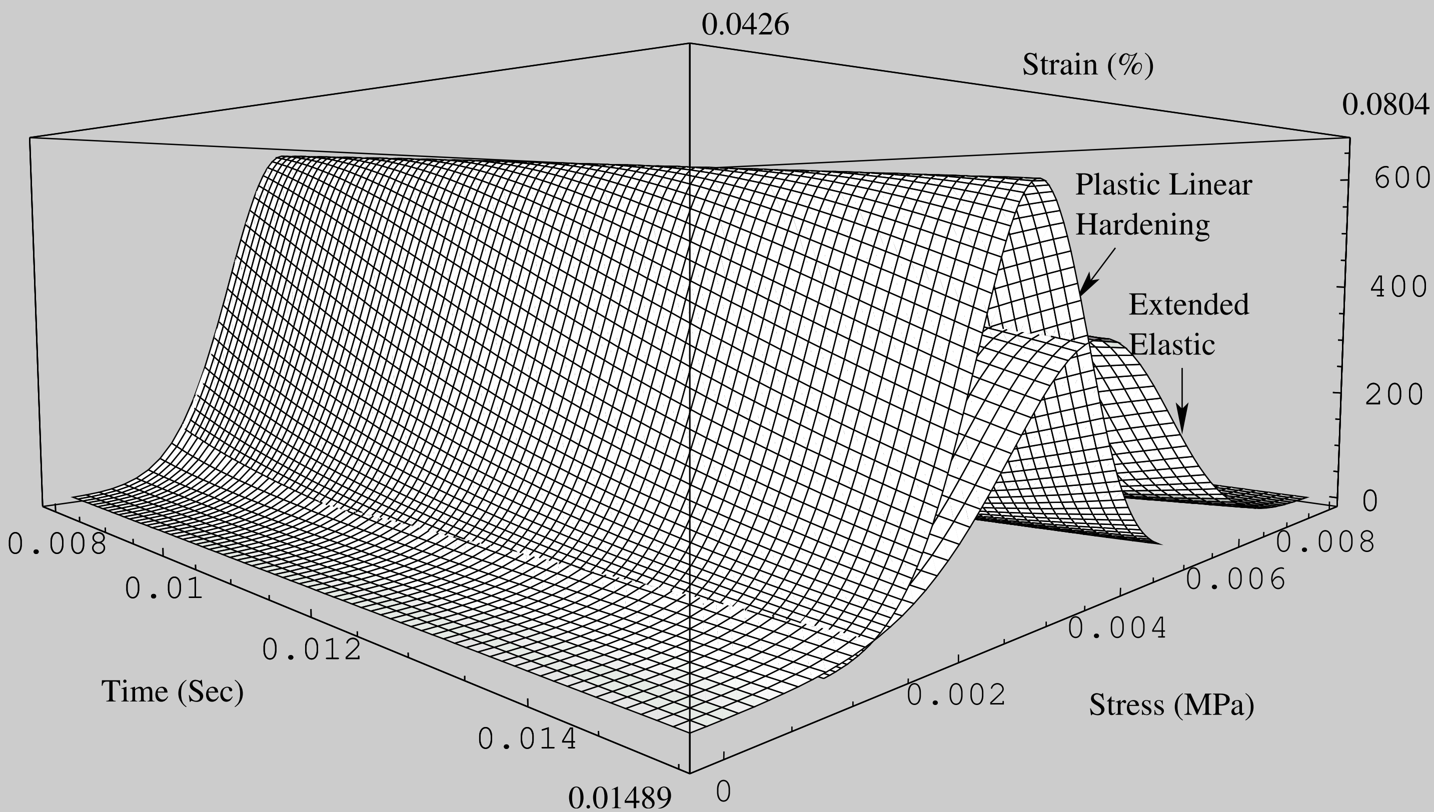
Drucker-Prager Associative Linear Hardening with Random G







Comparing Uncertainty in Elastic and Elastic–Plastic Response



Cam Clay Constitutive Model

- The general form of Cam Clay 1-D shear constitutive rate equation

$$
\begin{aligned}
\frac{d\sigma_{12}}{dt} &= G^{ep}\frac{d\epsilon_{12}}{dt} \\
&= \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)
\end{aligned}
$$

where η has the form:

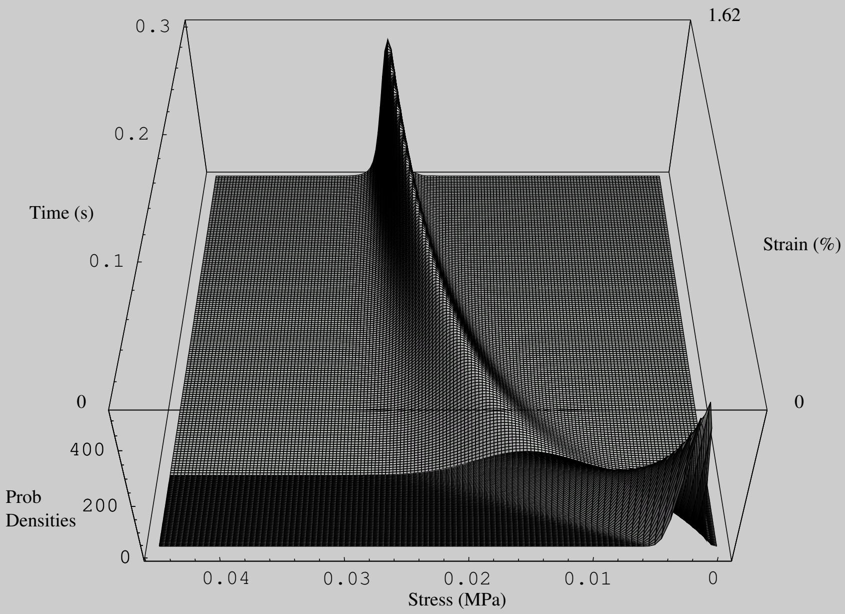
$$
\eta = \left[2G - \frac{\left(36\dfrac{G^2}{M^4}\right)\sigma_{12}^2}{\dfrac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\dfrac{G}{M^4}\right)\sigma_{12}^2 + \dfrac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)} \right] \frac{d\epsilon_{12}}{dt}
$$

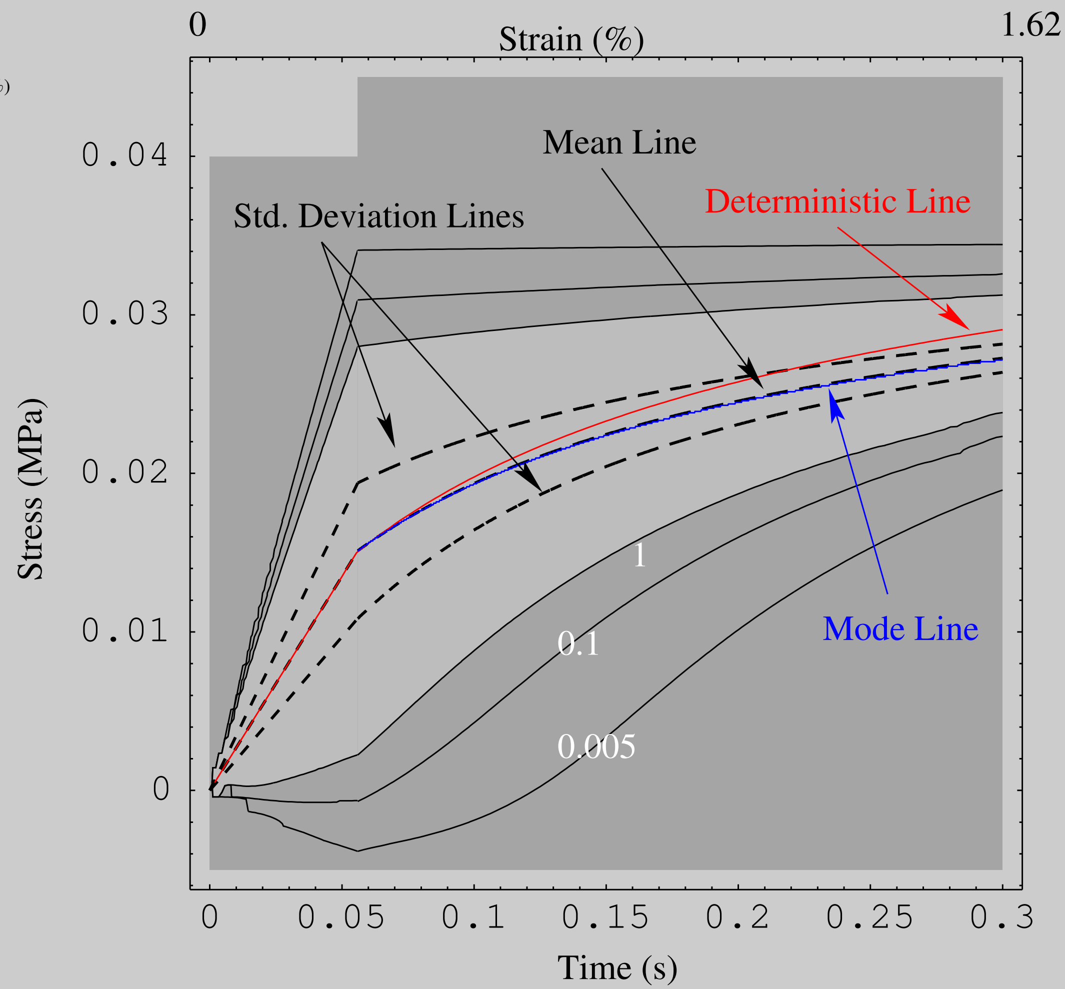
- The advection and diffusion coefficients of FPE are

$$
N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov \left[\frac{\partial \eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau) \right]
$$

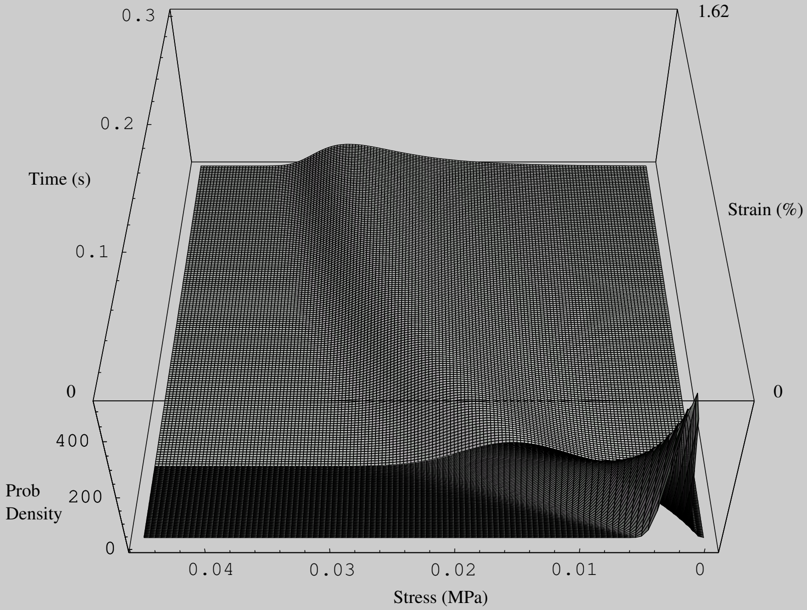
$$
N_{(2)}^{(i)} = \int_0^t d\tau cov \left[\eta^{(i)}(t); \eta^{(i)}(t-\tau) \right]
$$

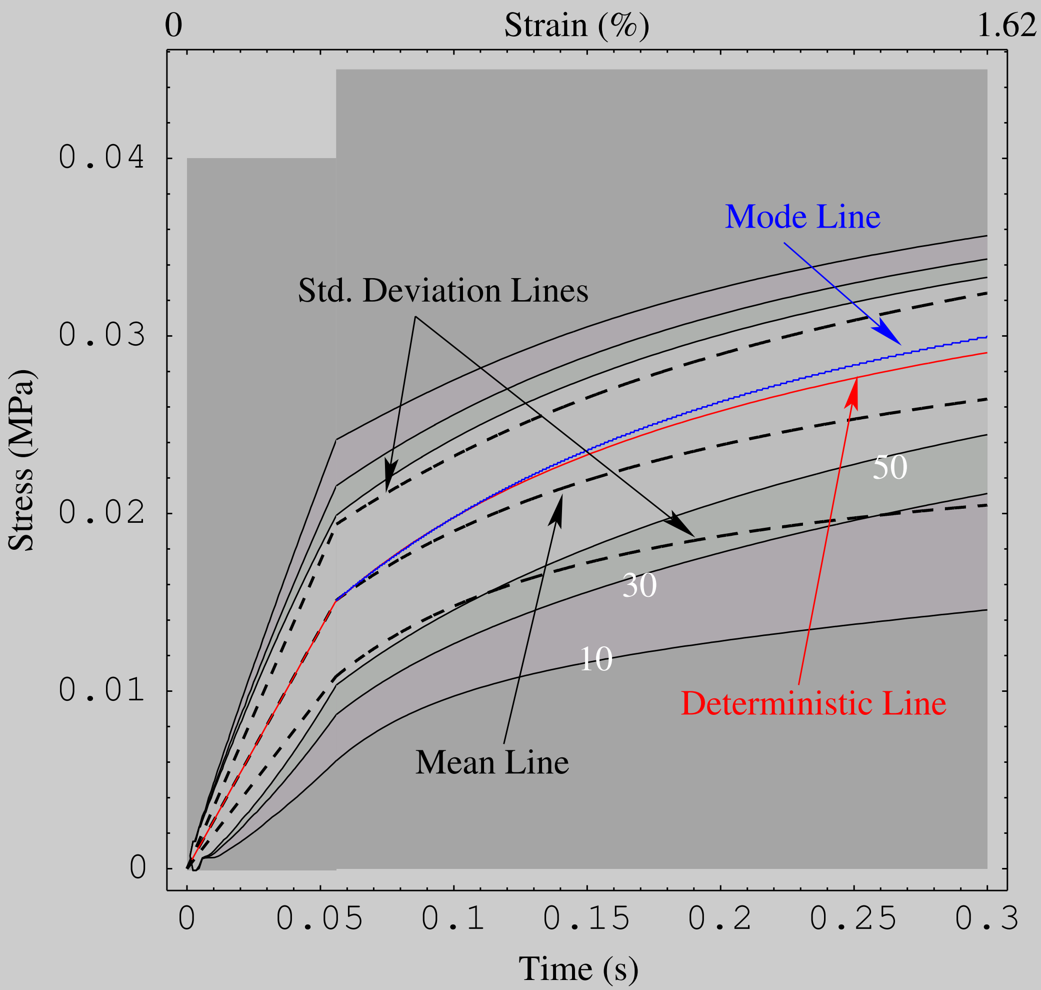
Low OCR Cam Clay with Random G



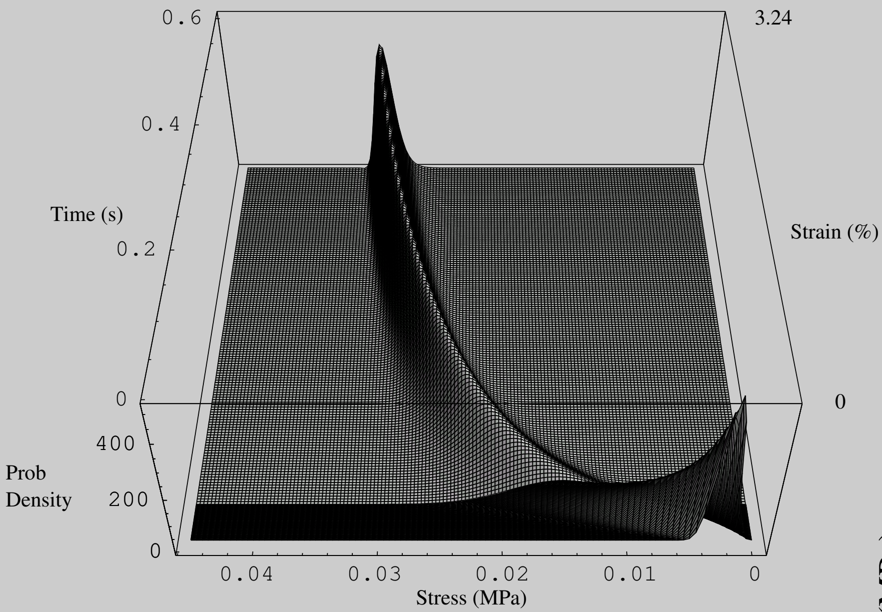


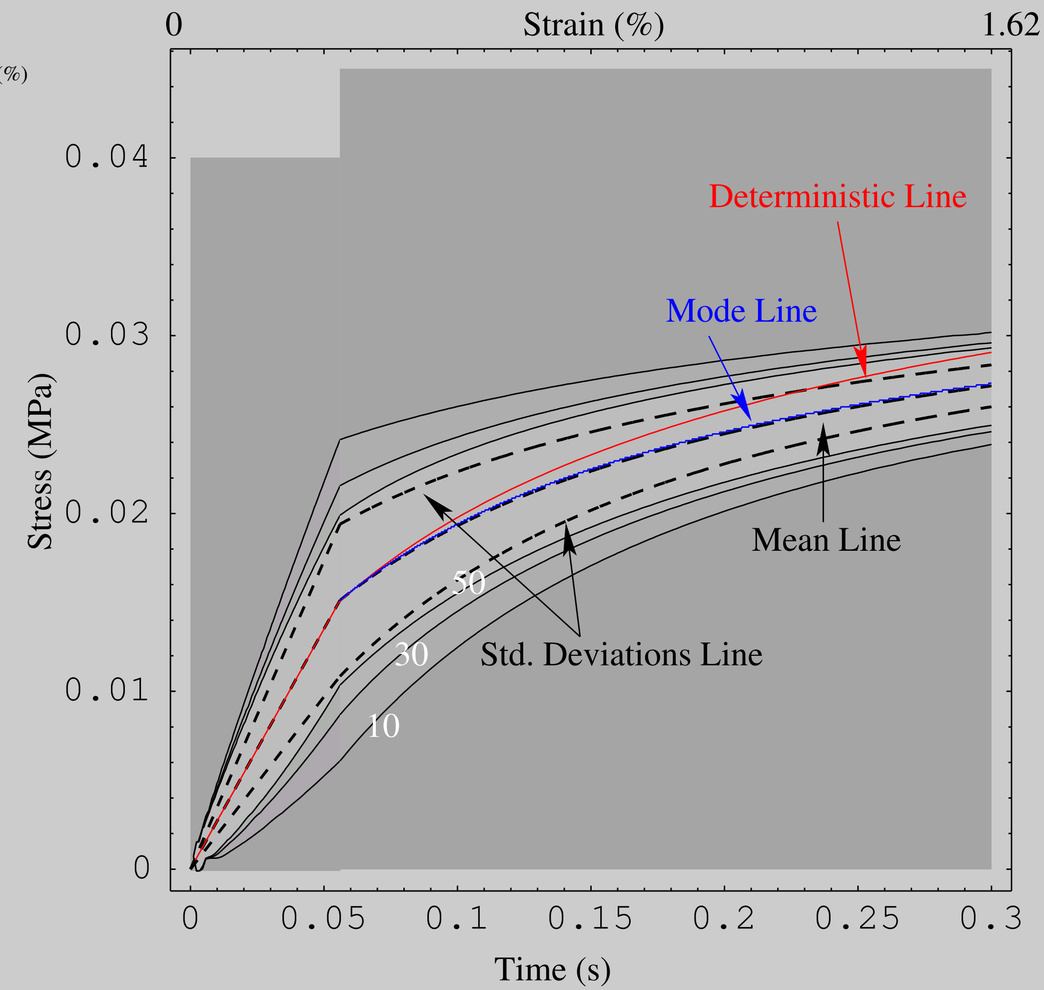
Low OCR Cam Clay Response with Random G and Random M



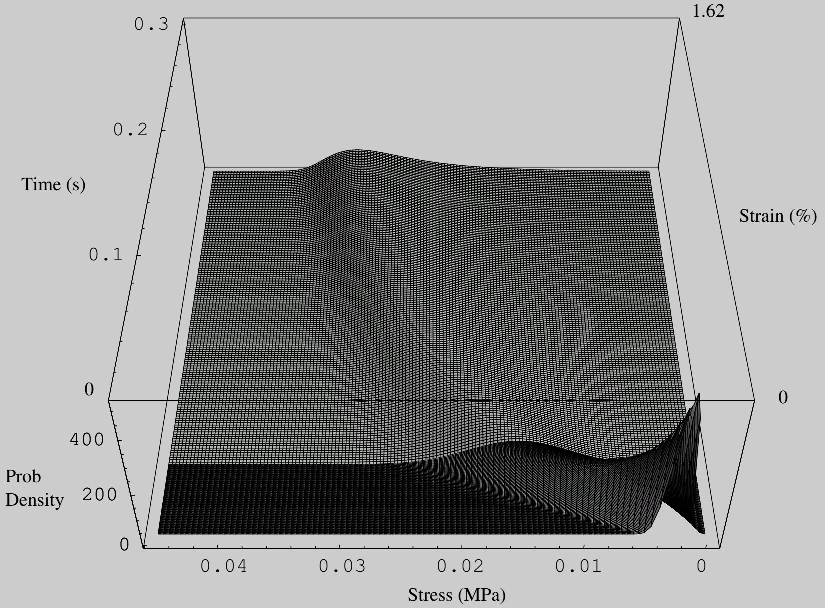


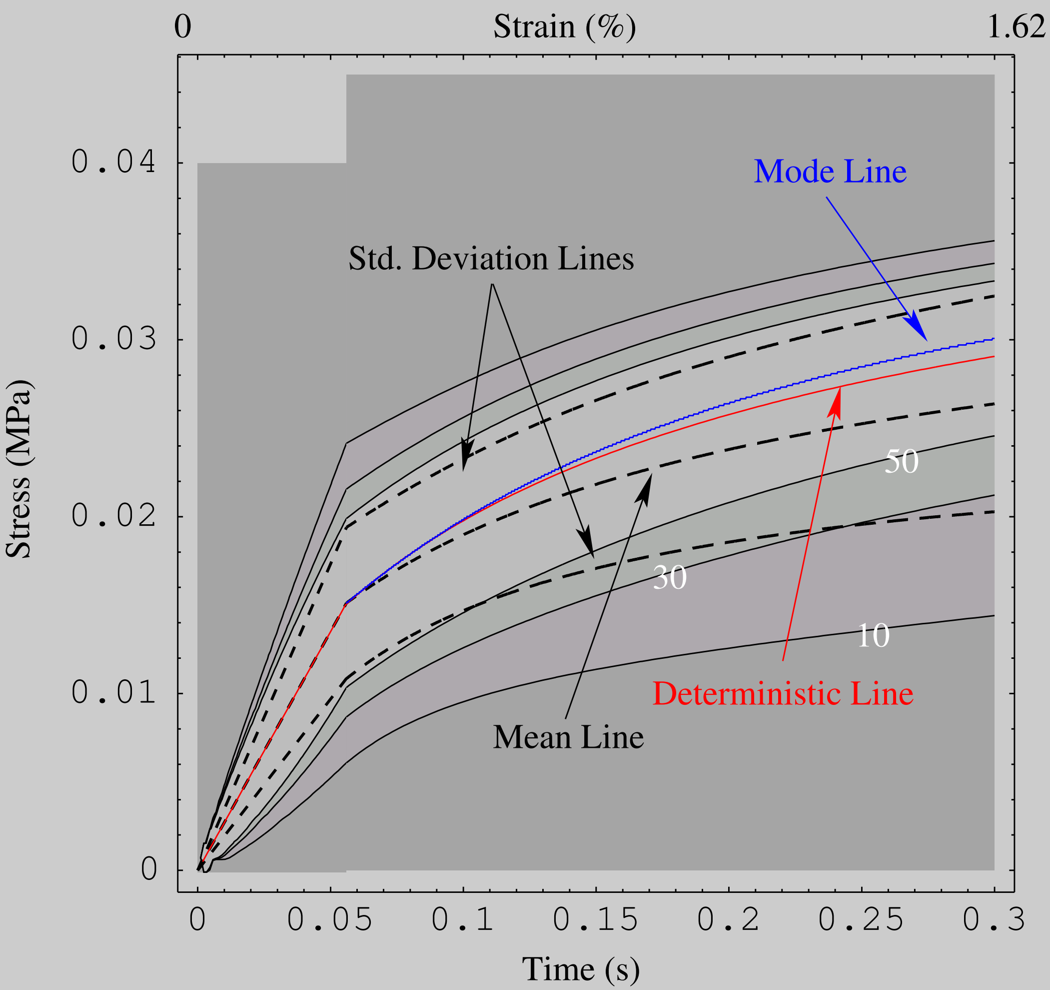
Low OCR Cam Clay Response with Random G and Random p_0



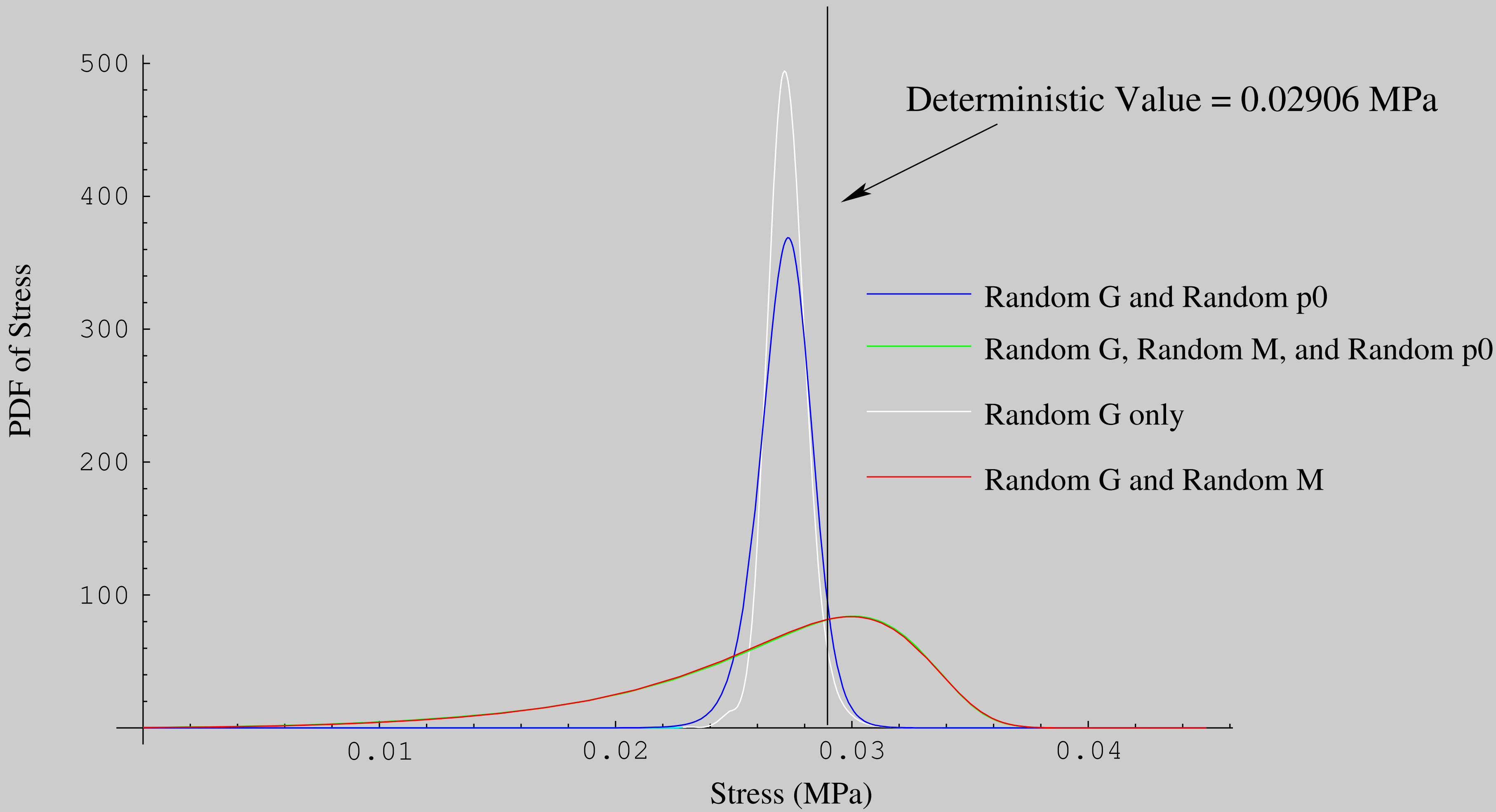


Low OCR Cam Clay Response with Random G, Random M and Random p_0

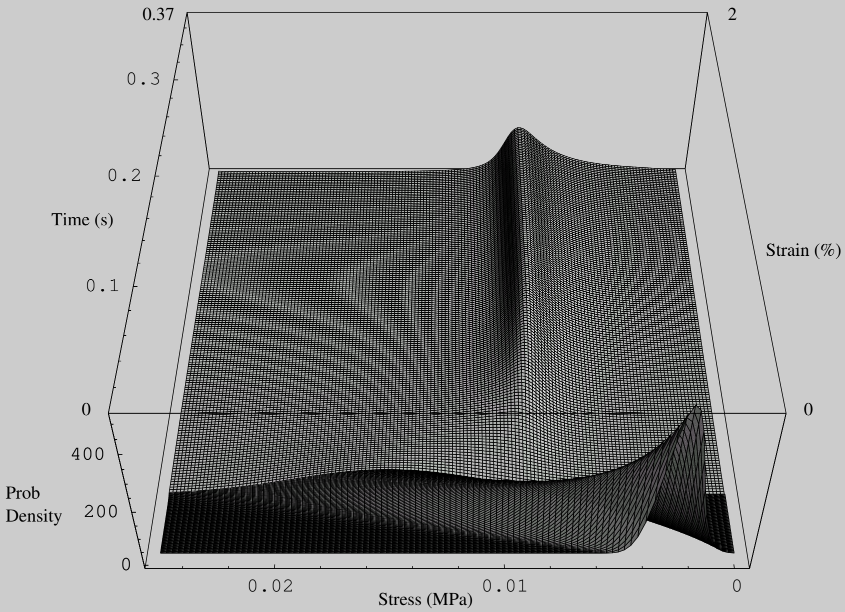


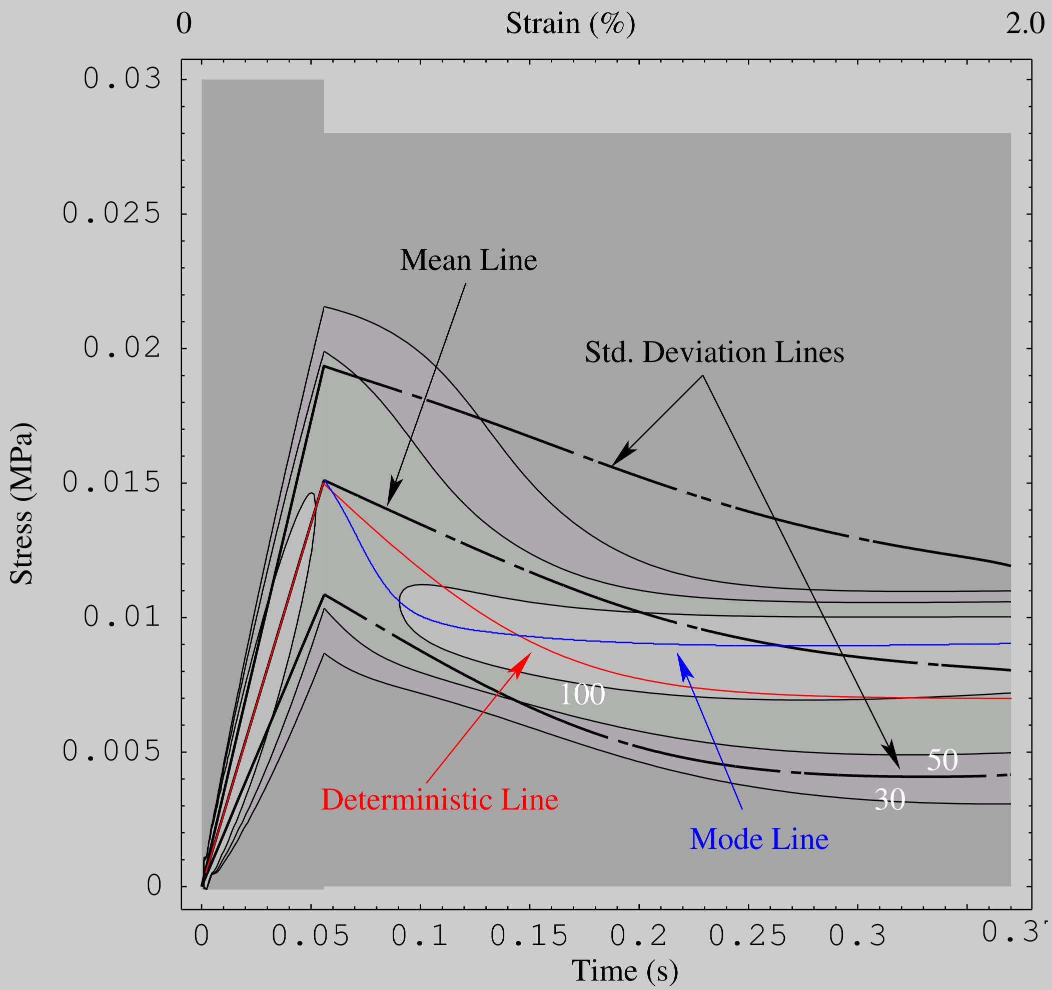


Low OCR Cam Clay Predictions at $\epsilon = 1.62$ %



High OCR Cam Clay Response with Random G and Random M





Summary

- Expression for evolution of probability densities of stress was derived for any general 1-D elastic-plastic constitutive rate equation.

- This method doesn't require repetitive use of computationally expensive deterministic elastic-plastic model and doesn't suffer from 'closure problem' associated with regular perturbation approach.

- Furthermore, the developed expression is linear and deterministic PDE whereas the constitutive rate equation is random and non-linear.

- Current work is going on in extending this method to 3-D and incorporating it to the formulation of stochastic elastic-plastic finite element method.