Uncertain Material Parameters and the Stress–Strain Response

Boris Jeremić

Department of Civil and Environmental Engineering
University of California, Davis

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Motivation

- Material behavior is uncertain, both spatially (stochastic) and point-wise (probabilistic),

- How is failure mechanics of solids and structures affected by non-uniformity (stochasticity, probability?)

- Can the Stochastic (Probabilistic) approach to Elasto-Plasticity offer more information about the failure of a particular solid?

- Can the Stochastic (Probabilistic) approach to Elasto-Plasticity offer more information (missing link) about the failure of general solids (and structures)
Motivation: Typical Soil Profile

![Soil Profile Diagram](image-url)
Previous Work

- Linear algebraic relations (linear elastic) \(\rightarrow\) analytical expressions:
  - variable transformation (Montgomery and Runger 2003)
  - cumulant expansion method (Gardiner 2004)

- Nonlinear differential equations
  - Monte Carlo analysis (Schueller 1997, De Lima et al., 2001, Mellah et al., 2000, Griffiths et al., 2005...)
Objectives of the Proposed Method

• Overcome the disadvantages of the perturbation and Monte Carlo approaches,

• Capable of carrying out sensitivity analysis at a point–location scale, when material parameter are modeled as random variables,

• Obtain probabilistic behavior of spatial average form (upscaled form) of constitutive rate equation when material properties are modeled as random field.
• The general 3-D constitutive rate equation - a nonlinear ODE system with random coefficient and random forcing

\[
\frac{d\sigma_{ij}(t)}{dt} = D_{ijkl} \frac{d\epsilon_{kl}(t)}{dt}
\]

\[
D_{ijkl} = \begin{cases} 
D_{ijkl}^{el} & \text{when } f < 0 \vee (f = 0 \land df < 0) \\
D_{ijkl}^{el} - \frac{D_{ijmn}^{el} \frac{\partial U}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} D_{pqkl}^{el}}{\frac{\partial f}{\partial \sigma_{rs}} D_{rstu}^{el} \frac{\partial U}{\partial \sigma_{tu}} - \frac{\partial f}{\partial q^*_r} r_*} & \text{when } f = 0 \vee df = 0
\end{cases}
\]
Stochastic Continuity Equation in 1D

• 1-D – a nonlinear ODE, random coefficient and random forcing

\[ \frac{d\sigma(t)}{dt} = \beta(\sigma, D, q, r; t) \frac{d\epsilon(t)}{dt} = \eta(\sigma, D, q, r, \epsilon; t) \]

with an initial condition \( \sigma(0) = \sigma_0 \)

• Consider a cloud of initial points (described by density \( \rho(\sigma, 0) \) in \( \sigma \)-space): movement of all these points is dictated by the constitutive equation, the phase density \( \rho \) varies in time according to a continuity equation (Liouville equation):

\[ \frac{\partial \rho(\sigma(t), t)}{\partial t} = -\frac{\partial}{\partial \sigma} \eta[\sigma(t), D, q, r, \epsilon(t)] \cdot \rho[\sigma(t), t] \]

with initial condition \( \rho(\sigma, 0) = \delta(\sigma - \sigma_0) \)
Fokker-Planck Equation

- Writing the continuity equation in ensemble average form and using Van Kampen’s Lemma (\( < \rho(h, t) > = P(h, t) \)) yields the following Fokker-Planck equation:

\[
\frac{\partial P(\sigma(t), t)}{\partial t} = - \frac{\partial}{\partial \sigma} \left\{ \left\langle \eta(\sigma(t), D, q, r, \epsilon(t)) \right\rangle \right. \\
+ \int_0^t d\tau Cov_0 \left[ \frac{\partial \eta(\sigma(t), D, q, r, \epsilon(t))}{\partial \sigma} \right] \eta(\sigma(t - \tau), D, q, r, \epsilon(t)) P(\sigma(t), t) \right\} \\
+ \frac{\partial^2}{\partial \sigma^2} \left\{ \int_0^t d\tau Cov_0 \left[ \eta(\sigma(t), D, q, r, \epsilon(t)) \right] \right. \\
\left. \quad \eta(\sigma(t - \tau), D, q, r, \epsilon(t)) \right\} P(\sigma(t), t) 
\]
Solution of Fokker-Planck Equation

- The Fokker-Planck equation $\rightarrow$ advection-diffusion equation:

$$\frac{\partial P(\sigma, t)}{\partial t} = -\frac{\partial}{\partial \sigma} \left[ N(1)P(\sigma, t) - \frac{\partial}{\partial \sigma} \{N(2)P(\sigma, t)\} \right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition – deterministic (Dirac delta function) or random

$$P(\sigma, 0) = \delta(\sigma)$$

- Boundary condition – reflecting (conserve probability mass or no probability current flow)

$$\zeta(\sigma, t)|_{At\, \text{Boundaries}} = 0$$

- The Fokker-Planck equation solution $\rightarrow$ Finite Difference Technique
Elastic Response with Random $G$

- General form of elastic constitutive rate equation
  \[ \frac{d\sigma_{12}}{dt} = 2G\frac{d\epsilon_{12}}{dt} = \eta(G, \epsilon_{12}; t) \]

- The advection and diffusion coefficients of FPE are
  \[ N_{(1)} = 2\frac{d\epsilon_{12}}{dt} < G > \quad ; \quad N_{(2)} = 4t (\frac{d\epsilon_{12}}{dt})^2 Var[G] \]
Cam Clay Constitutive Model

- The general form of Cam Clay 1-D shear constitutive rate equation

\[
\frac{d\sigma_{12}}{dt} = G^{ep} \frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)
\]

where \(\eta\) has the form:

\[
\eta = \left[ 2G - \frac{(1 + e_0)p(2p - p_0)^2}{\kappa} + \left( \frac{36 G^2}{M^4} \right) \frac{\sigma_{12}^2}{\kappa} + \left( \frac{18 G}{M^4} \right) \frac{\sigma_{12}^2}{\kappa} + \frac{1 + e_0}{\lambda - \kappa} pp_0 (2p - p_0) \right] \frac{d\epsilon_{12}}{dt}
\]

- The advection and diffusion coefficients of FPE are

\[
N^{(i)}_{(1)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau \text{cov} \left[ \frac{\partial \eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t - \tau) \right]
\]

\[
N^{(i)}_{(2)} = \int_0^t d\tau \text{cov} \left[ \eta^{(i)}(t); \eta^{(i)}(t - \tau) \right]
\]
Low OCR Cam Clay with Random $G$
Low OCR Cam Clay, Random $G$ and $M$
Low OCR Cam Clay, Random $G$ and $\rho_0$
Low OCR Cam Clay, Random $G$, $M$ and $\rho_0$

![Graphical representation of stress, strain, time, and probability density for Low OCR Cam Clay, Random $G$, $M$ and $\rho_0$](image)
Low OCR Cam Clay Predictions at $\epsilon = 1.62\%$

Deterministic Value = 0.02906 MPa
High OCR Cam Clay, Random $G$ and $M$

![Graph showing stress, strain, and time relationships with probabilistic density contours and mean line, along with deterministic and mode lines.](image)

Boris Jeremić, UC Davis
Summary

• Expression (linear and deterministic PDE) for evolution of probability densities of stress available for any general elastic-plastic constitutive rate equation (random and non-linear).

• Propagation of uncertainty (any symmetric of nonsymmetric distribution) in material parameters on stress strain response.

• Short term work in progress, simulation formulation (FEM: Polynomial chaos expansion, Karhunen–Loeve expansion) for spatial quantification of non-uniformity in soils and influence on response and failure.

• Long term goal: given requirement for probabilistic performance assessment, can we prescribe in–situ and lab. test program.