Uncertain Material Parameters and the Stress–Strain Response

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Motivation

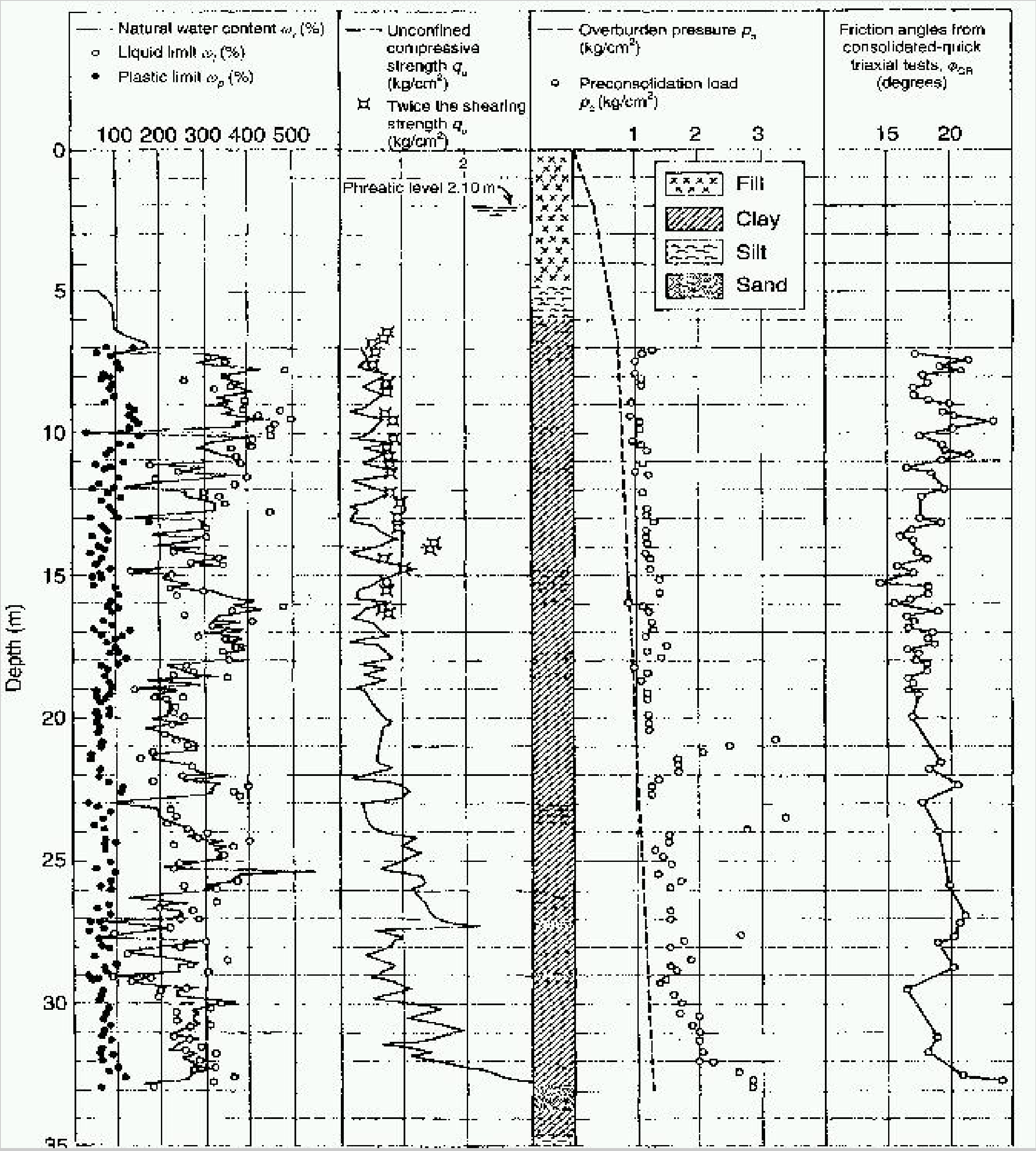
- Material behavior is uncertain, both spatially (stochastic) and point-wise (probabilistic),

- How is failure mechanics of solids and structures affected by non–uniformity (stochasticity, probability?

- Can the Stochastic (Probabilistic) approach to Elasto–Plasticity offer more information about the failure of a *particular* solid?

- Can the Stochastic (Probabilistic) approach to Elasto–Plasticity offer more information (missing link) about the failure of *general* solids (and structures)

Motivation: Typical Soil Profile



Previous Work

- Linear algebraic relations (linear elastic) $\rightarrow$ analytical expressions:
  - variable transformation (Montgomery and Runger 2003)
  - cumulant expansion method (Gardiner 2004)

- Nonlinear differential equations $\rightarrow$
  - Monte Carlo analysis (Schueller 1997, De Lima et al, 2001, Mellah et al. 2000, Griffiths et al. 2005...)
  - Perturbation approach (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 2997, Mellah et al, 2000)

Objectives of the Proposed Method

- Overcome the disadvantages of the perturbation and Monte Carlo approaches,
- Capable of carrying out sensitivity analysis at a point–location scale, when material parameter are modeled as random variables,
- Obtain probabilistic behavior of spatial average form (upscaled form) of constitutive rate equation when material properties are modeled as random field.

Problem Statement

- The general 3-D constitutive rate equation - a nonlinear ODE system with random coefficient and random forcing

$$\frac{d\sigma_{ij}(t)}{dt} = D_{ijkl}\frac{d\epsilon_{kl}(t)}{dt}$$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{when } f < 0 \vee (f = 0 \wedge df < 0) \\ \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn}\dfrac{\partial U}{\partial \sigma_{mn}}\dfrac{\partial f}{\partial \sigma_{pq}}D^{el}_{pqkl}}{\dfrac{\partial f}{\partial \sigma_{rs}}D^{el}_{rstu}\dfrac{\partial U}{\partial \sigma_{tu}} - \dfrac{\partial f}{\partial q_*}r_*} & \text{when } f = 0 \vee df = 0 \end{cases}$$

Stochastic Continuity Equation in 1D

- 1-D – a nonlinear ODE, random coefficient and random forcing

$$\frac{d\sigma(t)}{dt} = \beta(\sigma, D, q, r; t)\frac{d\epsilon(t)}{dt} = \eta(\sigma, D, q, r, \epsilon; t)$$

with an initial condition $\sigma(0) = \sigma_0$

- Consider a cloud of initial points (described by density $\rho(\sigma, 0)$ in σ-space): movement of all these points is dictated by the constitutive equation, the phase density ρ varies in time according to a continuity equation (Liouville equation):

$$\frac{\partial \rho(\sigma(t), t)}{\partial t} = -\frac{\partial}{\partial \sigma}\eta[\sigma(t), D, q, r, \epsilon(t)].\rho[\sigma(t), t]$$

with initial condition $\rho(\sigma, 0) = \delta(\sigma - \sigma_0)$

Fokker-Planck Equation

- Writing the continuity equation in ensemble average form and using Van Kampen's Lemma ($< \rho(h,t) >= P(h,t)$) yields the following Fokker-Planck equation:

$$
\begin{aligned}
\frac{\partial P(\sigma(t),t)}{\partial t} = & - \frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(t),D,q,r,\epsilon(t))\right\rangle \right.\right. \\
& + \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(t),D,q,r,\epsilon(t))}{\partial \sigma};\right. \\
& \left.\left.\left.\eta(\sigma(t-\tau),D,q,r,\epsilon(t-\tau)\right]\right\} P(\sigma(t),t)\right] \\
& + \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(t),D,q,r,\epsilon(t));\right.\right.\right. \\
& \left.\left.\left.\eta(\sigma(t-\tau),D,q,r,\epsilon(t-\tau))\right]\right\} P(\sigma(t),t)\right]
\end{aligned}
$$

Solution of Fokker-Planck Equation

- The Fokker-Planck equation $\rightarrow$ advection-diffusion equation:

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition – deterministic (Dirac delta function) or random

$$P(\sigma,0) = \delta(\sigma)$$

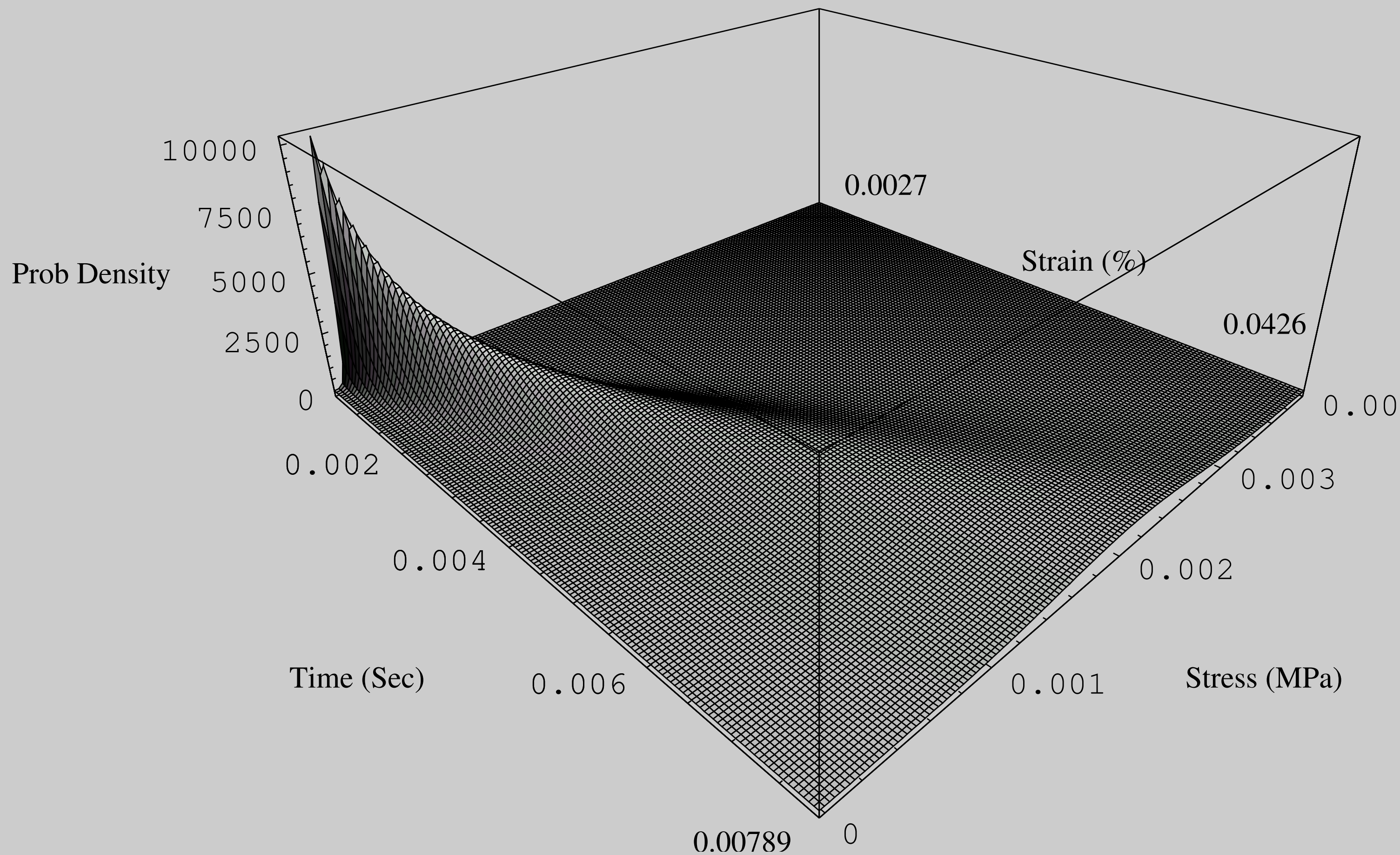
- Boundary condition – reflecting (conserve probability mass or no probability current flow)

$$\zeta(\sigma,t)|_{AtBoundaries} = 0$$

- The Fokker-Planck equation solution $\rightarrow$ *Finite Difference Technique*

Elastic Response with Random G

- General form of elastic constitutive rate equation
  $d\sigma_{12}/dt = 2Gd\epsilon_{12}/dt = \eta(G, \epsilon_{12}; t)$
- The advection and diffusion coefficients of FPE are
  $N_{(1)} = 2d\epsilon_{12}/dt < G >$; $N_{(2)} = 4t\,(d\epsilon_{12}/dt)^2\,Var[G]$



Cam Clay Constitutive Model

- The general form of Cam Clay 1-D shear constitutive rate equation

$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, D^{el}, q, r, \epsilon_{12}; t)$$

where η has the form:

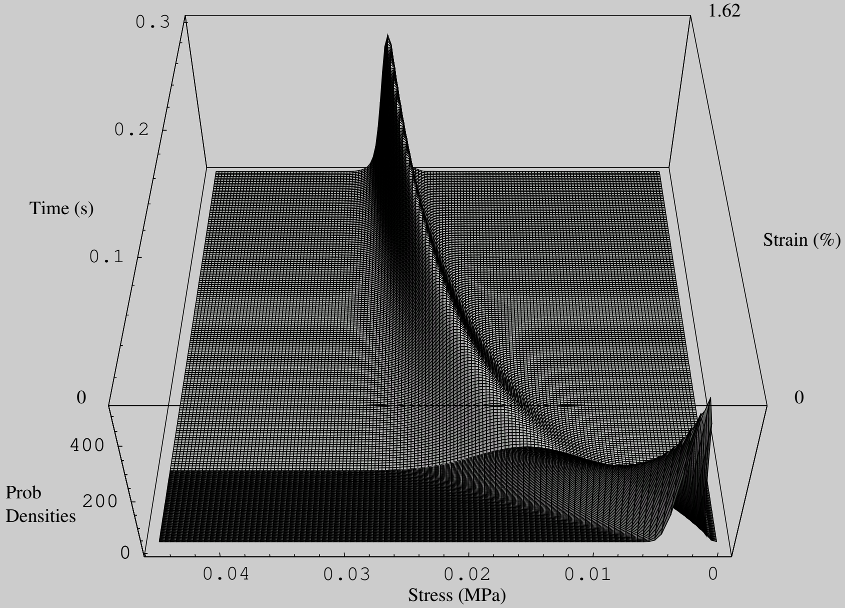
$$\eta = \left[2G - \frac{\left(36\frac{G^2}{M^4}\right)\sigma_{12}^2}{\frac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\frac{G}{M^4}\right)\sigma_{12}^2 + \frac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)} \right] \frac{d\epsilon_{12}}{dt}$$

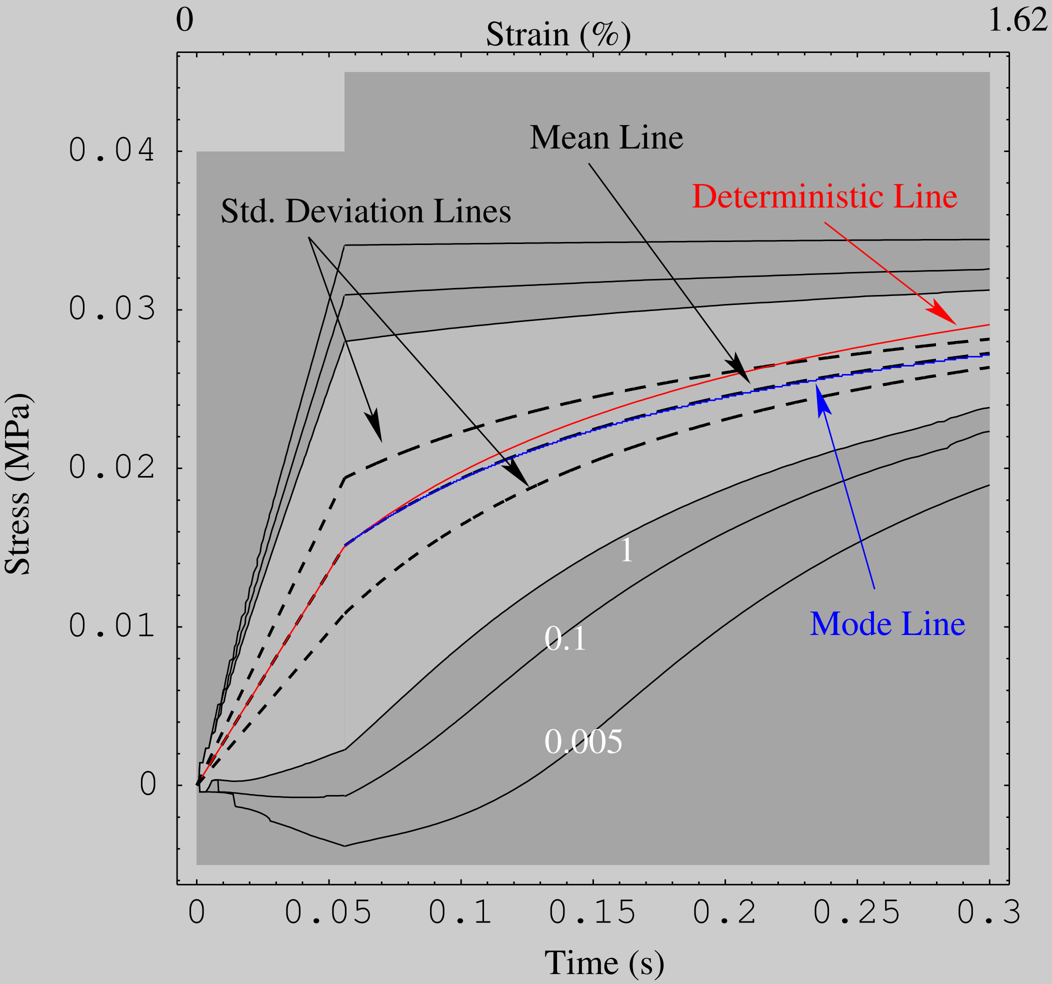
- The advection and diffusion coefficients of FPE are

$$N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov\left[\frac{\partial \eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau)\right]$$

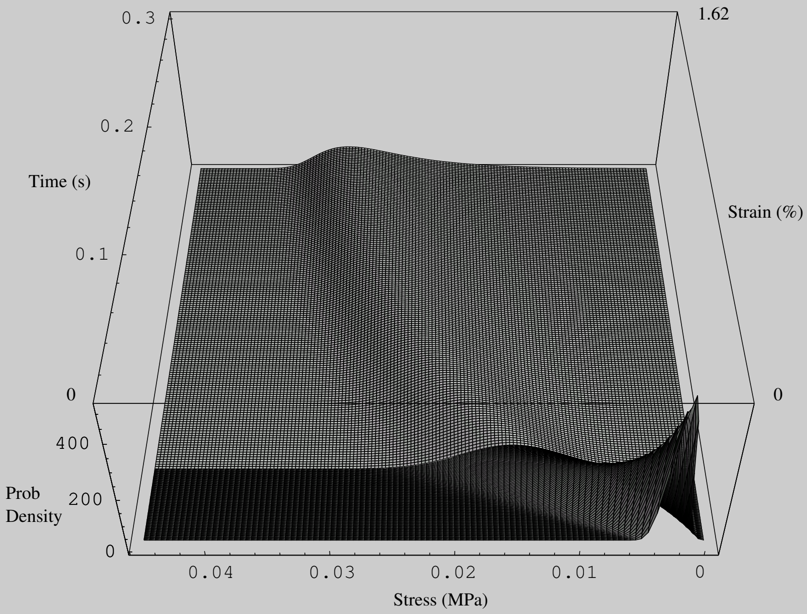
$$N_{(2)}^{(i)} = \int_0^t d\tau cov\left[\eta^{(i)}(t); \eta^{(i)}(t-\tau)\right]$$

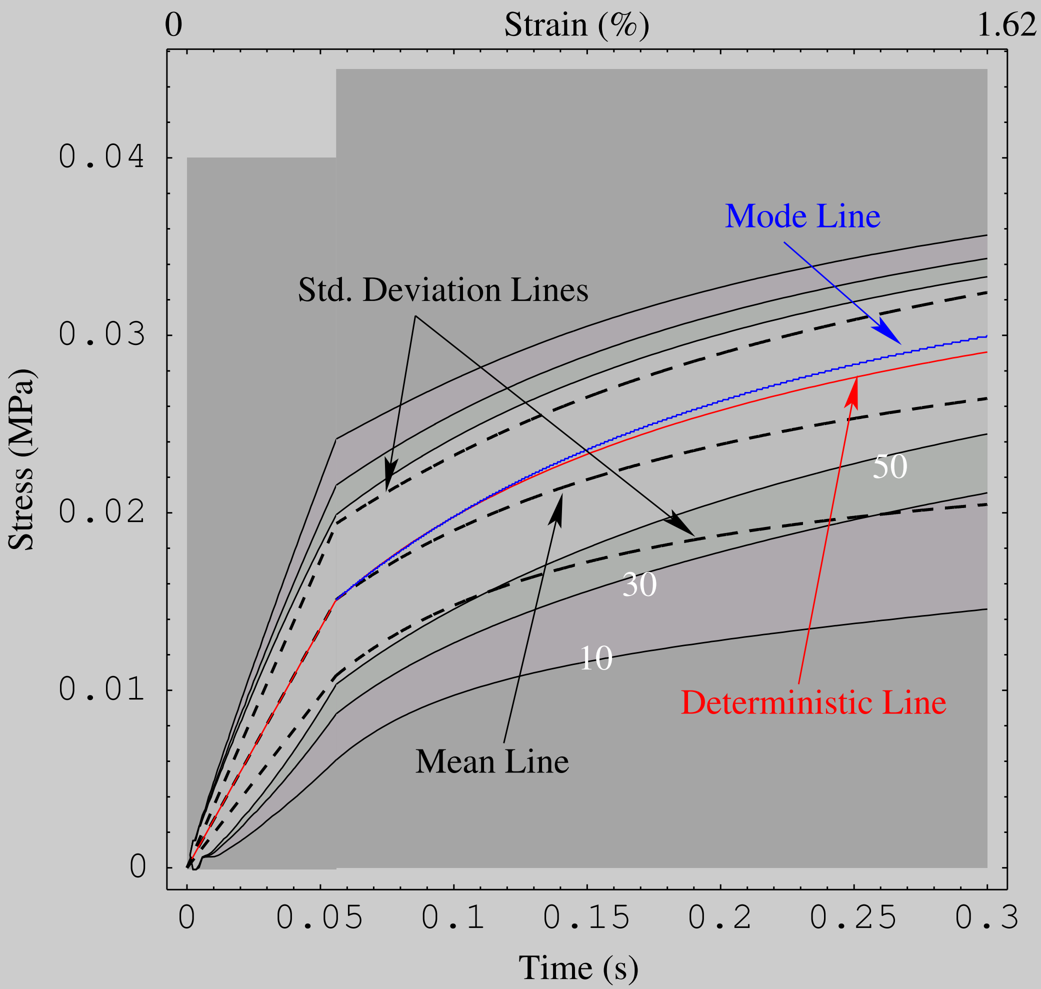
Low OCR Cam Clay with Random G



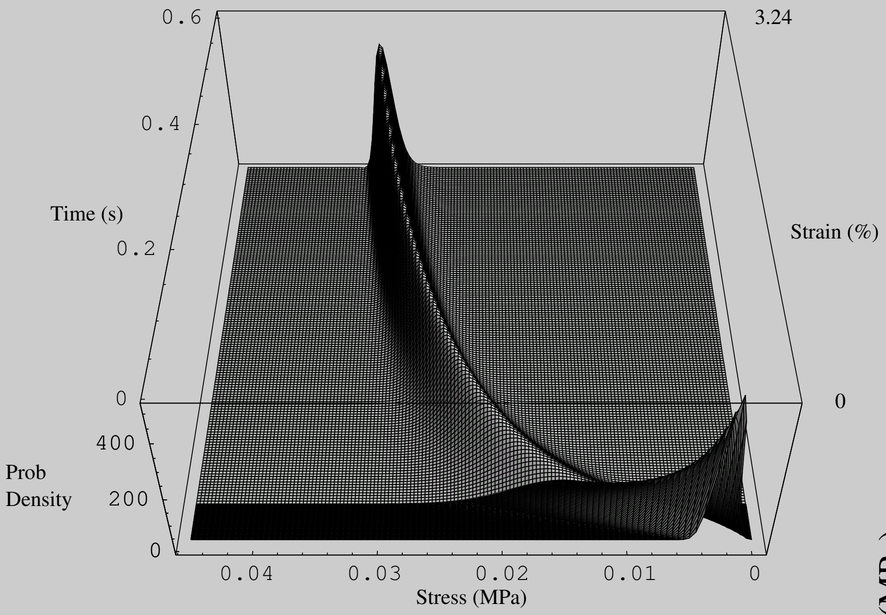


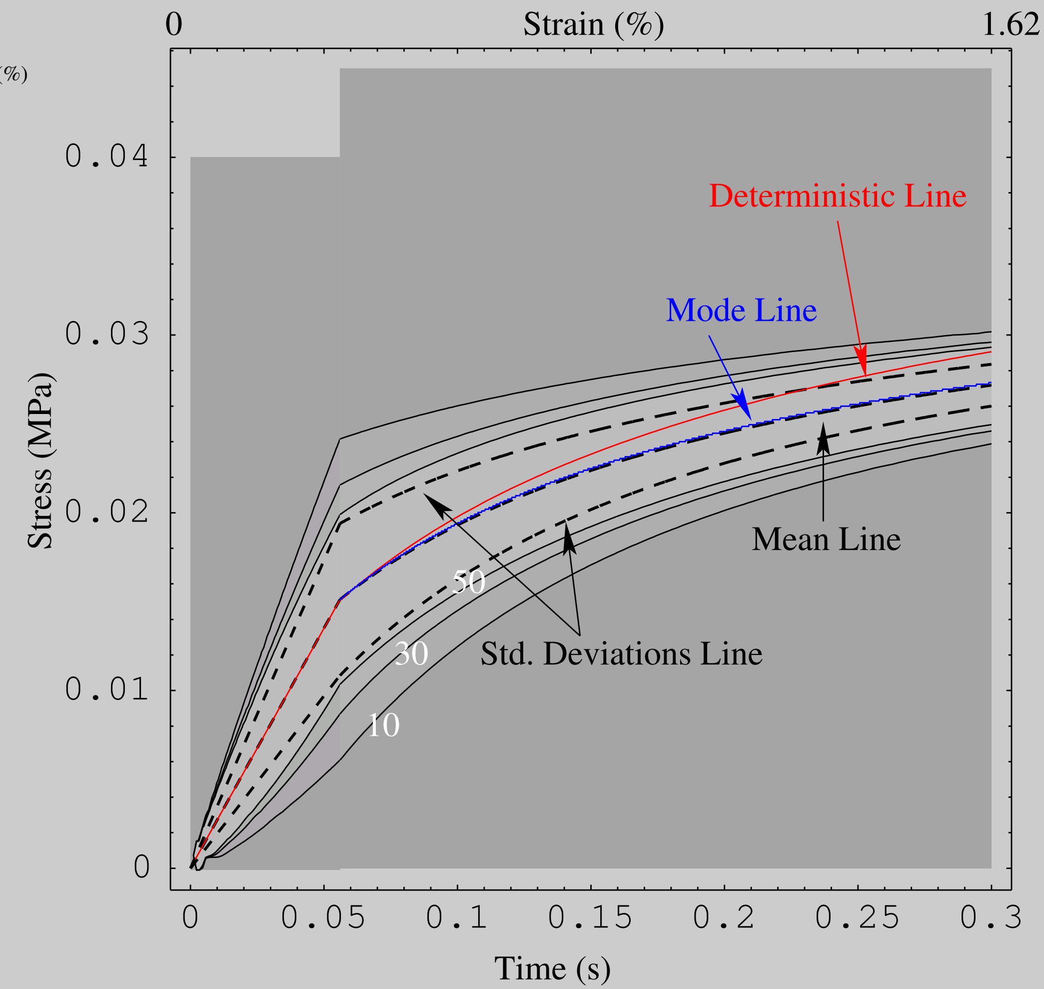
Low OCR Cam Clay, Random G and M



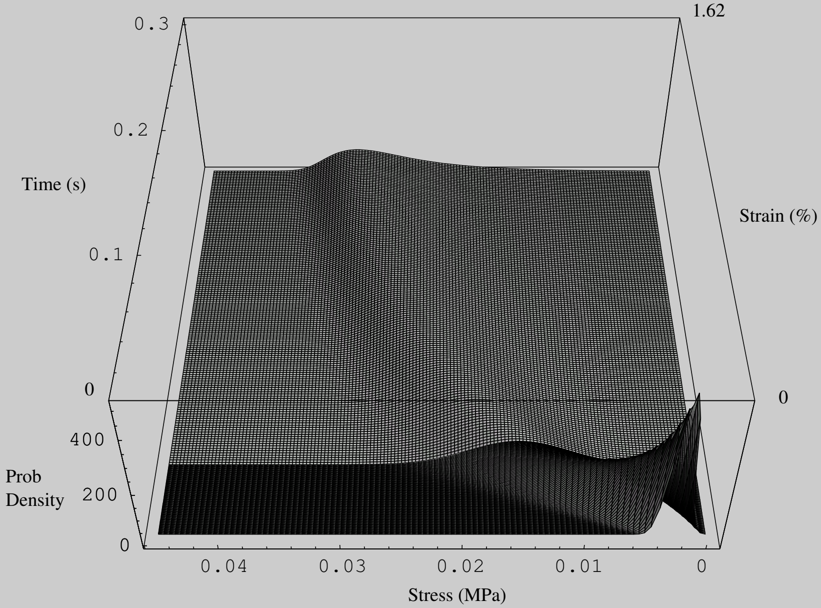


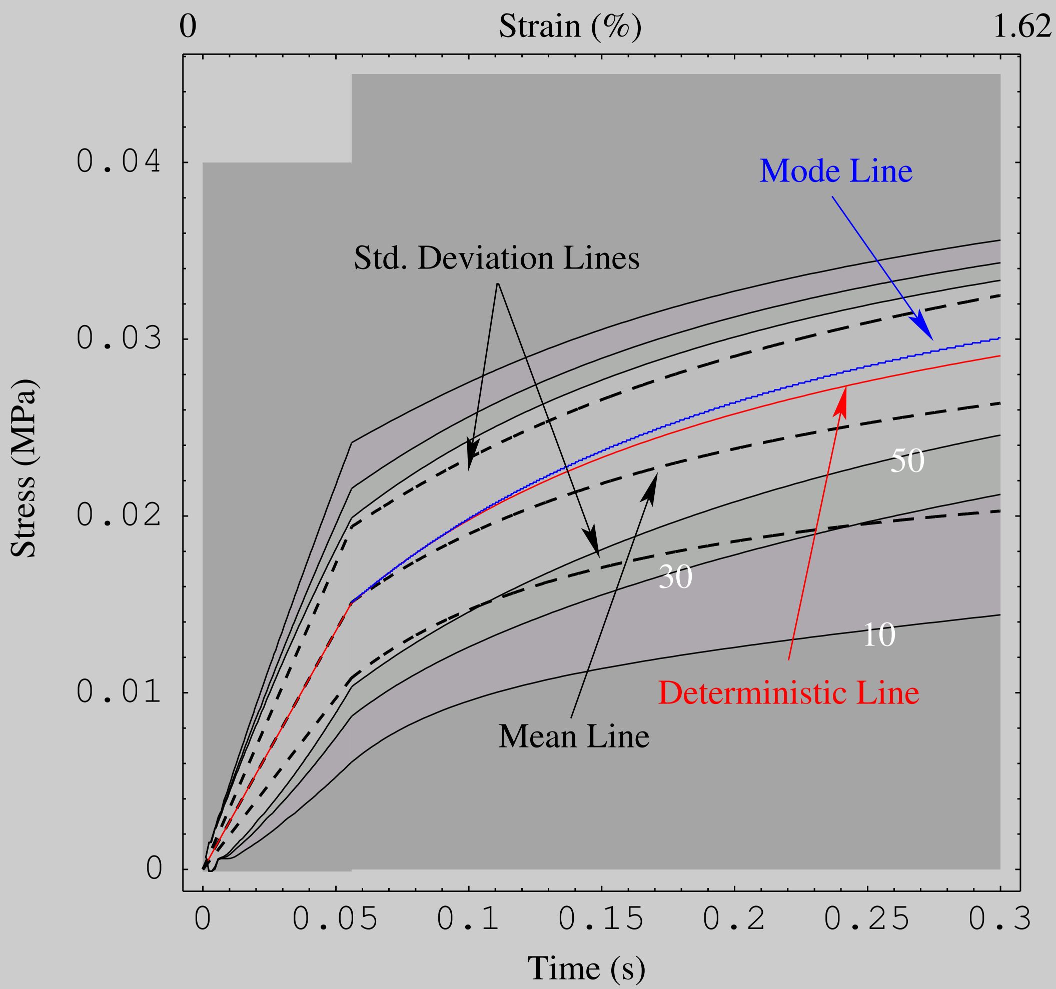
Low OCR Cam Clay, Random G and p_0



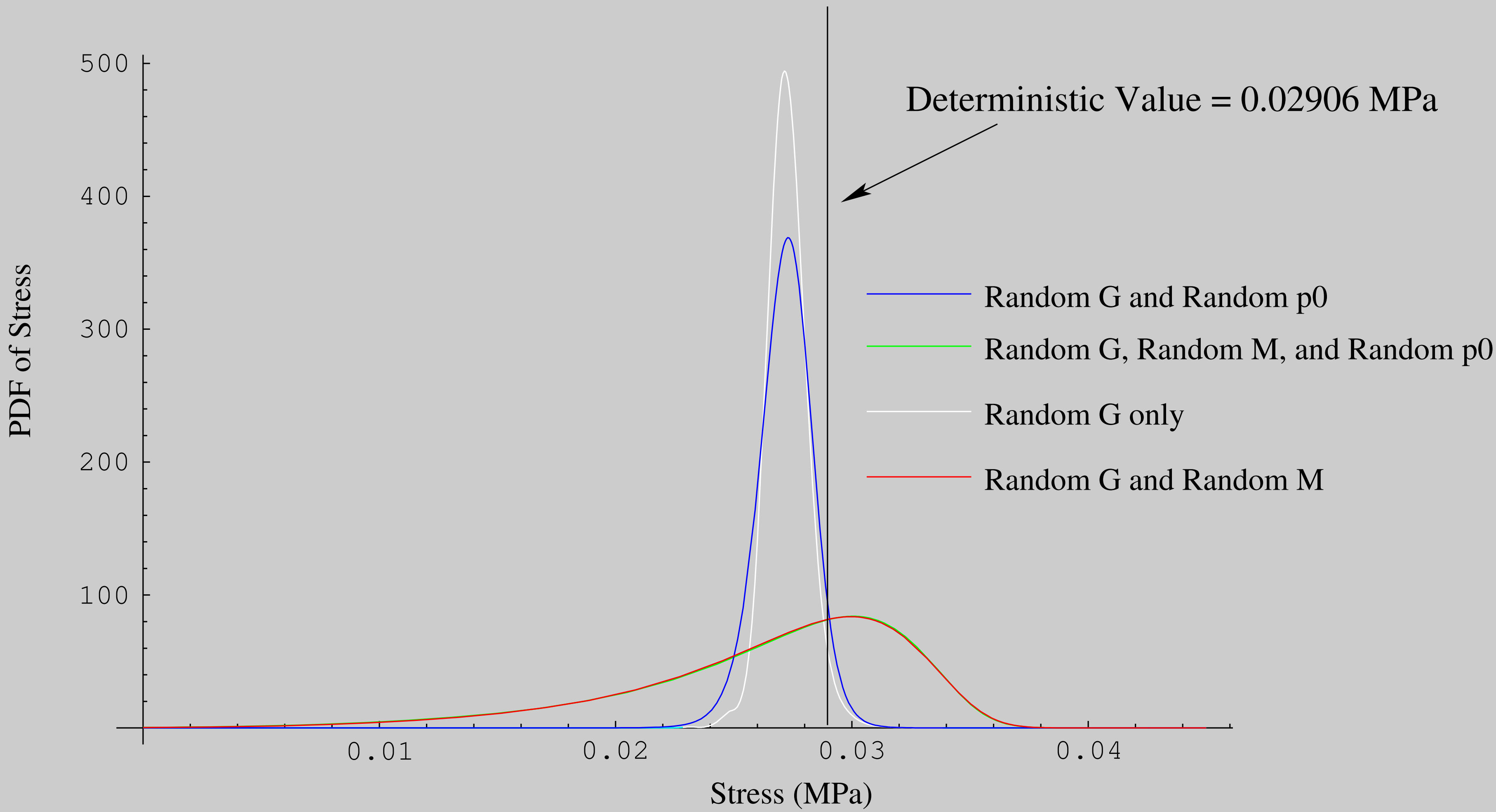


Low OCR Cam Clay, Random G, M and p_0

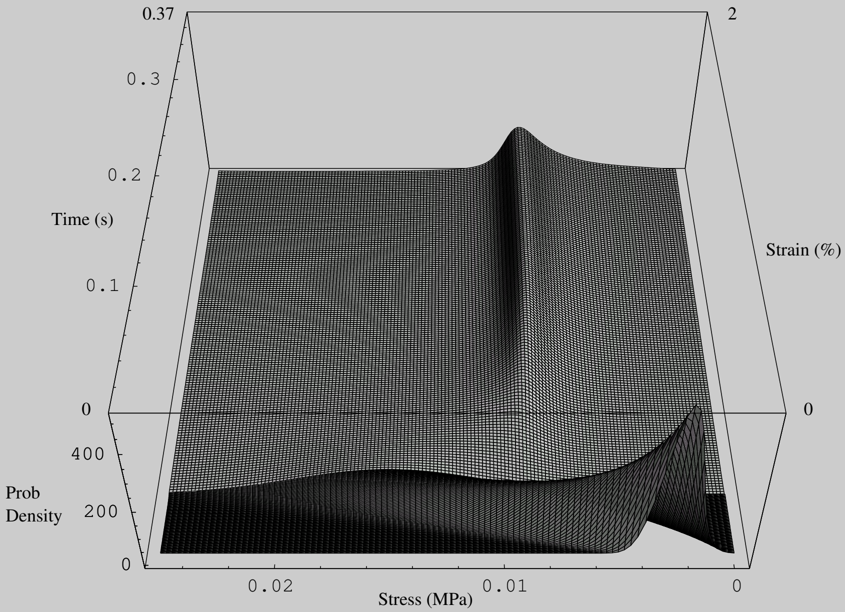


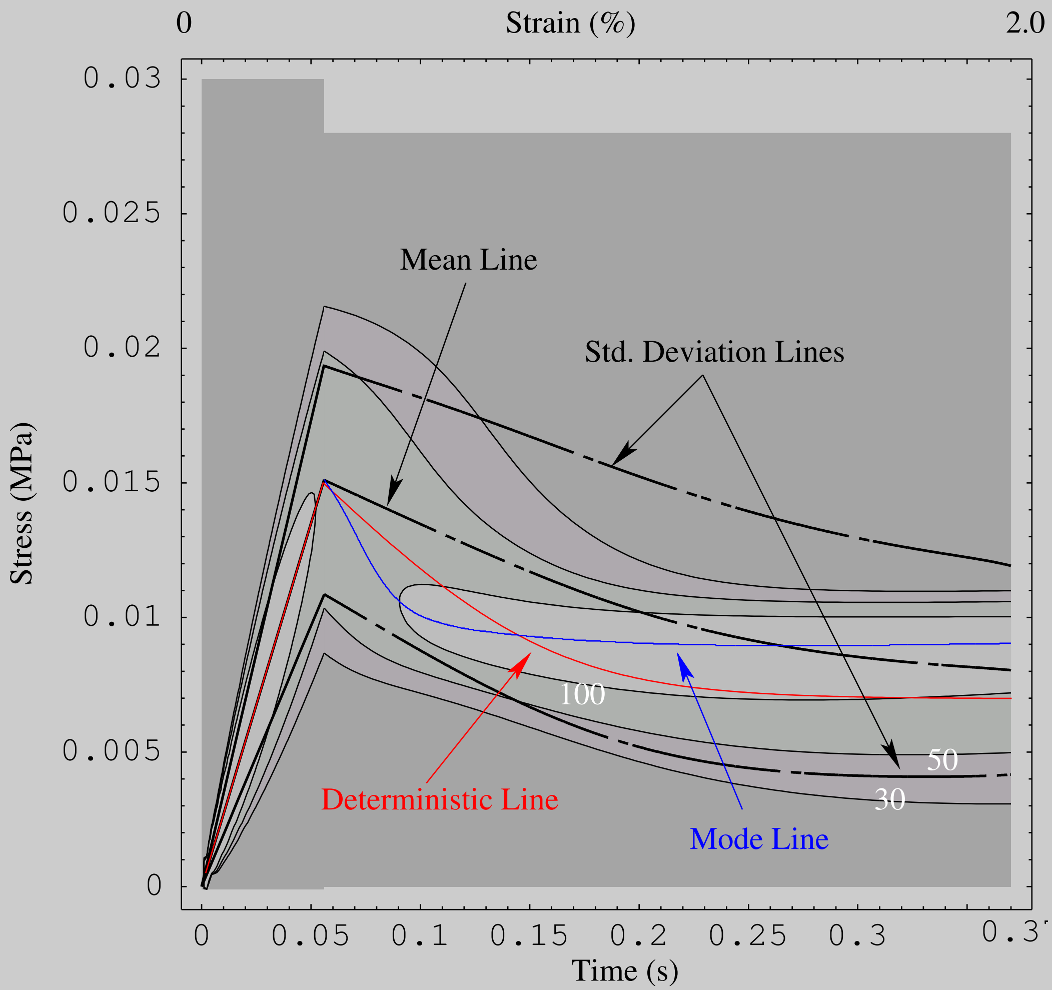


Low OCR Cam Clay Predictions at $\epsilon = 1.62$ %



High OCR Cam Clay, Random G and M





Summary

- Expression (linear and deterministic PDE) for evolution of probability densities of stress available for any general elastic-plastic constitutive rate equation (random and non-linear).

- Propagation of uncertainty (any symmetric of nonsymmetric distribution) in material parameters on stress strain response.

- Short term work in progress, simulation formulation (FEM: Polynomial chaos expansion, Karhunen–Loeve expansion) for spatial quantification of non–uniformity in soils and influence on response and failure.

- Long term goal: given requirement for probabilistic performance assessment, can we prescribe in–situ and lab. test program.