The Role of Material Variability and Uncertainty in Elastic-Plastic Finite Element Simulations

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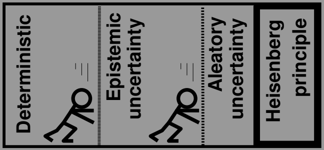
Outline

- Motivation
  - Historical Overview
  - Uncertainties in Material
- Boundary Value Problem
  - Propagation of Uncertainties in Mechanics (Geomechanics)
  - Stochastic Finite Element Method
- Probabilistic Elasto–Plasticity
  - Probabilistic Elastic–Plastic: Differential Equation
  - Probabilistic Elastic–Plastic Response

Outline

Types of Uncertainties

- Epistemic uncertainty - uncertainty due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools (neural network, fuzzy logic etc.) are not well developed → trade-off with aleatory uncertainty
- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools (classical second-order analysis) to deal with



Brownian Motions

- Governing (Langevin) equation:

$$m\frac{dv}{dt} = F(x) - \beta v + \eta(t)$$

- Probability density function (PDF) of particle displacement obeys a simple diffusion equation (Einstein (1905)):

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2}$$

- Addition of external forces (gravity, elastic or magnetic attraction) $\rightarrow$ Fokker-Planck-Kolmogorov (FPK) equation governs the PDF (Kolmogorov 1941)
- Alternately, Monte Carlo method can be used for solution of Langevin equation $\rightarrow$ computationally very expensive

Stochastic Systems: Random Forcing

- Classical approach: relationship between the autocorrelation function and spectral density function (Wiener 1930) $\rightarrow$ Paved the way to the solution of general stochastic differential equation (SDE)
- SDEs with random forcing $\rightarrow$ Highly developed mathematical theory for Itô type equation:

$$dx = a(x,t)dt + b(x,t)dW$$

  - Solution is a Markov process
  - PDF of solution process satisfies a FPK PDE

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[a(x,t)p(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b^2(x,t)p(x,t)\right]$$

Stochastic Systems: Random Coefficient

- Approximate Solution methods
  - Functional integration approach (Hopf 1952)
  - Averaged equation approach (Bharrucha-Reid 1968)
  - Numerical approaches
  - Monte Carlo method

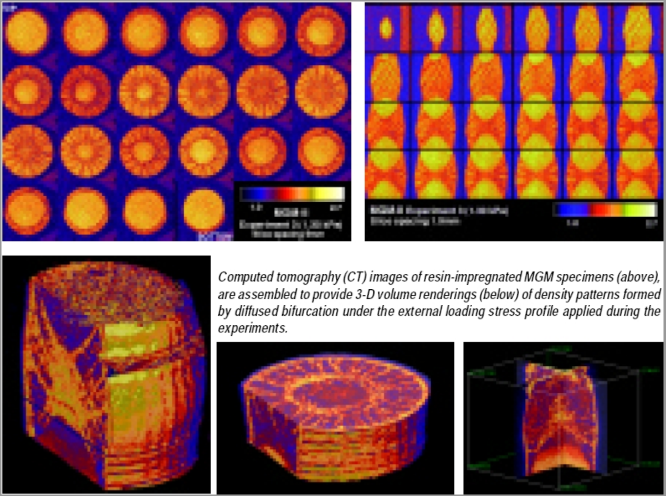
- FPK equation for the characteristic functional of the solution for problem of wave propagation in random media (Lee 1974)

- Eulerian-Lagrangian form of FPK equation for probabilistic solution of flow through porous media (Kavvas 2003)

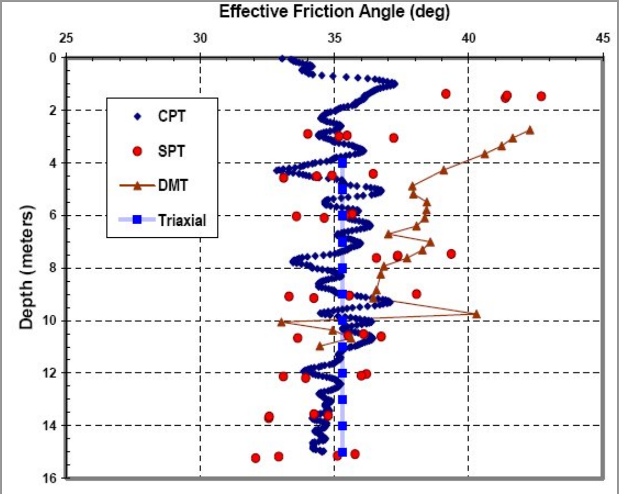
Material Uncertainties

- Material's (concrete, metals, soil, rock, bone, foam, powder etc.) behavior is inherently uncertain
  - Spatial variability
  - Point-wise uncertainty - testing error, transformation error
- Failure mechanisms related to spatial variability (strain localization and bifurcation of response)
- Inverse problems
  - New material design, (*point-wise*)
  - Solid and/or structure design (or retrofits), (*spatial*)

Soil: Inside Failure (MGM)



Soil: Spatial Variation (Mayne et al. (2000))



Soil Uncertainties and Quantifications

- Natural variability of soil deposit (Fenton 1999) $\rightarrow$ function of soil formation process
- Testing error (Stokoe et al. 2004)
  - Imperfection of instruments
  - Error in methods to register quantities
- Transformation error (Phoon and Kulhawy 1999)
  - Correlation by empirical data fitting (e.g. CPT data $\rightarrow$ friction angle etc.)

Probabilistic material (Soil Site) Characterization

- Ideal: complete probabilistic site characterization
- Large (physically large but not statistically) amount of data
  - Site specific mean and coefficient of variation (COV)
  - Covariance structure from similar sites (e.g. Fenton 1999)
- Minimal data: general guidelines for typical sites and test methods (Phoon and Kulhawy (1999))
  - COVs and covariance structures of inherent variabilities
  - COVs of testing errors and transformation uncertainties.
- Moderate amount of data $\rightarrow$ Bayesian updating (e.g. Phoon and Kulhawy 1999, Baecher and Christian 2003)

Outline

Propagation of Uncertainties in Mechanics

Governing equation

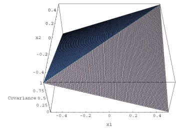
- Dynamic problems $\rightarrow M\ddot{u} + C\dot{u} + Ku = \phi$
- Static problems $\rightarrow Ku = \phi$

Existing solution methods

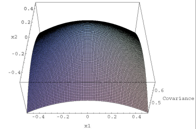
- **Random r.h.s** (external force random)
  - FPK equation approach
  - Use of fragility curves with deterministic FEM
- **Random l.h.s** (material properties random)
  - Monte Carlo approach with DFEM $\rightarrow$ CPU expensive
  - Stochastic finite element method (Perturbation method, fails if COVs of soil $>$ 20% ; Spectral method $\rightarrow$ elastic material)

Truncated Karhunen–Loeve Expansion for Input Field

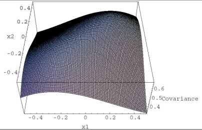
- Input random fields represented in eigen-modes of covariance kernel
- Error minimizing property
- Minimizes number of stochastic dimensions



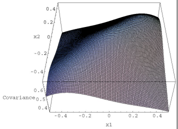
Exact cov. surface



1-term



2-terms



3-terms

Model covariance kernel: $C(x_1, x_2) = e^{-|x_1 - x_2|/b}$

Truncated K-L approximation: $C(x_1, x_2) = \sum_{k=1}^{M} \lambda_k f_k(x_1) f_k(x_2)$

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Polynomial Chaos (PC) Expansion for DOFs

- DOF covariance kernel is not known a priori (unknown eigenvalues e_j and eigenvectors $b_j(x)$)
  $u(x,\theta) = \sum_{j=1}^{L} e_j \;\; \chi_j(\theta) \;\; b_j(x)$
- DOFs expressed as functionals of known input random variables and unknown deterministic function
  $u(x,\theta) = \zeta[\xi_i(\theta), x]$
- Need a basis of known random variables $\rightarrow$ PC expansion
  $\chi_j(\theta) = \sum_{i=0}^{P} \gamma_i^{(j)} \psi_i \left[\{\xi_r\}\right];$
  $u(x,\theta) = \sum_{j=1}^{L} \sum_{i=0}^{P} \gamma_i^{(j)} \psi_i[\{\xi_r\}] e_j b_j(x) = \sum_{i=0}^{P} \psi_i[\{\xi_r\}] d_i(x)$
- Deterministic coefficients can be found by minimizing norm of error of finite representation (e.g. using Galerkin scheme)

SSEPFEM Formulation

$$\sum_{n=1}^{N} K_{mn} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} C_{ijk} K'_{mnk} = \langle F_m \psi_i[\{\zeta_r\}] \rangle$$

$$K_{mn} = \int_D B_n D B_m dV \qquad K'_{mnk} = \int_D B_n \sqrt{\lambda}_k h_k B_m dV$$

$$C_{ijk} = \langle \zeta_k(\theta) \psi_i[\{\zeta_r\}] \psi_j[\{\zeta_r\}] \rangle \qquad F_m = \int_D \phi N_m dV$$

- Generalized DOF
- Material (soil) nonlinearity → Constitutive integration at Gauss point → Probabilistic Elasto–Plasticity

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Outline

Uncertainty Propagation through Constitutive Eq.

- General 3-D elastic-plastic constitutive law →
$$\frac{d\sigma_{ij}}{dt} = D_{ijkl}\frac{d\epsilon_{kl}}{dt}$$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{for elastic} \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn}\dfrac{\partial U}{\partial \sigma_{mn}}\dfrac{\partial f}{\partial \sigma_{pq}}D^{el}_{pqkl}}{\dfrac{\partial f}{\partial \sigma_{rs}}D^{el}_{rstu}\dfrac{\partial U}{\partial \sigma_{tu}} - \dfrac{\partial f}{\partial q_*}r_*} & \text{for elastic–plastic} \end{cases}$$

- Non–linear coupling in the coefficient (elastic–plastic modulus)
- Focusing on 1-D constitutive Behavior → a nonlinear ODE with random coefficient and random forcing

Previous Works

- Linear algebraic or differential equations → Analytical solution:
  - Variable Transformation Method (Montgomery and Runger 2003)
  - Cumulant Expansion Method (Gardiner 2004)
- Nonlinear differential equations (elasto-plastic/viscoelastic-viscoplastic):
  - Monte Carlo Simulation (Schueller 1997, De Lima et al 2001, Mellah et al. 2000, Griffiths et al. 2005...)
  - Perturbation Method (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 1997)
- Monte Carlo method: accurate, very costly
- Perturbation method: first and second order Taylor series expansion about mean - limited to problems having small C.O.V. and inherits 'closure problem'

Problem Statement

- General 3-D elastic-plastic constitutive law:

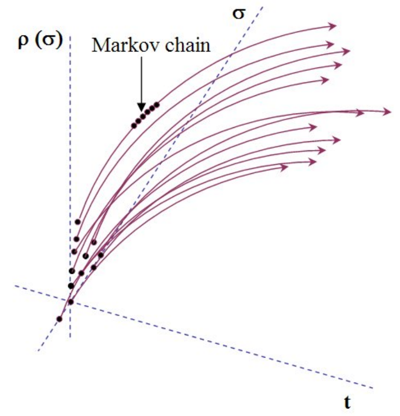
$$d\sigma_{ij} = \left\{ D^{el}_{ijkl} - \frac{D^{el}_{ijmn} \frac{\partial U}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} D^{el}_{pqkl}}{\frac{\partial f}{\partial \sigma_{rs}} D^{el}_{rstu} \frac{\partial U}{\partial \sigma_{tu}} - \frac{\partial f}{\partial q_*} r_*} \right\} d\epsilon_{kl}$$

- Focusing on 1-D constitutive Behavior $\rightarrow$ a nonlinear ODE with random coefficient and random forcing

$$\begin{aligned} \frac{d\sigma(x,t)}{dt} &= \beta(\sigma(x,t), D^{el}(x), q(x), r(x); x, t)\frac{d\epsilon(x,t)}{dt} \\ &= \eta(\sigma, D^{el}, q, r, \epsilon; x, t) \end{aligned}$$

with an initial condition $\sigma(0) = \sigma_0$

Stochastic Continuity (Liouville) Equation



$$\frac{\partial \rho(\sigma(x,t),t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[\eta(\sigma(x,t), D^{el}(x), q(x), r(x), \epsilon(x,t))\right]\rho[\sigma(x,t),t]$$

Initial condition:

$$\rho(\sigma,0) = \delta(\sigma - \sigma_0)$$

$\rho(\sigma,0) \rightarrow$ density of probabilistic solutions in σ space

Ensemble Average form of Liouville Equation

$\rightarrow$ van Kampen's Lemma $\rightarrow < \rho(\sigma, t) >= P(\sigma, t)$, ensemble average of phase density is the probability density;
$\rightarrow$ Continuity equation written in ensemble average form (eg. cumulant expansion method (Kavvas and Karakas 1996)):

$$\frac{\partial \langle \rho(\sigma(x_t, t), t) \rangle}{\partial t} = -\frac{\partial}{\partial \sigma} \left[\left\{ \left\langle \eta(\sigma(x_t, t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t, t)) \right\rangle \right. \right.$$

$$+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta(\sigma(x_t, t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t, t))}{\partial \sigma}; \right.$$

$$\left. \left. \left. \eta(\sigma(x_{t-\tau}, t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau) \right] \right\} \langle \rho(\sigma(x_t, t), t) \rangle \right]$$

$$+ \frac{\partial^2}{\partial \sigma^2} \left[\left\{ \int_0^t d\tau Cov_0 \left[\eta(\sigma(x_t, t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t, t)); \right. \right. \right.$$

$$\left. \left. \left. \eta(\sigma(x_{t-\tau}, t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)) \right] \right\} \langle \rho(\sigma(x_t, t), t) \rangle \right]$$

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Eulerian–Lagrangian FPK Equation

$$
\begin{aligned}
\frac{\partial P(\sigma(x_t,t),t)}{\partial t} &= -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t,t),D^{el}(x_t),q(x_t),r(x_t),\epsilon(x_t,t))\right\rangle\right.\right. \\
&+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta(\sigma(x_t,t),D^{el}(x_t),q(x_t),r(x_t),\epsilon(x_t,t))}{\partial\sigma};\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau),D^{el}(x_{t-\tau}),q(x_{t-\tau}),r(x_{t-\tau}),\epsilon(x_{t-\tau},t-\tau)\right]\right\} P(\sigma(x_t,t),t)\right] \\
&+ \frac{\partial^2}{\partial\sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(x_t,t),D^{el}(x_t),q(x_t),r(x_t),\epsilon(x_t,t));\right.\right.\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau),D^{el}(x_{t-\tau}),q(x_{t-\tau}),r(x_{t-\tau}),\epsilon(x_{t-\tau},t-\tau))\right]\right\} P(\sigma(x_t,t),t)\right]
\end{aligned}
$$

- Complete probabilistic description of response
- Second-order exact to covariance of time
- Deterministic equation (in probability density space)

Solution of FPK Equation

- FPK equation → advection-diffusion equation or continuity equation

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition
  - Deterministic → Dirac delta function → $P(\sigma,0) = \delta(\sigma)$
  - Random → Any given distribution
- Boundary condition: Reflecting BC → conserves probability mass $\zeta(\sigma,t)|_{At\ Boundaries} = 0$
- Numerical scheme → *Finite Difference Technique*

Application of FPK equation to Material Models

- FPK equation is applicable to any incremental elastic–plastic material model (only the coefficients $N_{(1)}$ and $N_{(2)}$ differ)
- Unique attributes of probabilistic solution
  - Solution in terms of PDF, not a single value of stress
  - Influence of initial condition on the PDF of stress
  - Transition between elastic and elastic–plastic
  - Symmetry and non–symmetry in PDF of stress
  - Differences in mean, mode and deterministic solution of stress
  - Interaction of random soil properties on the PDF of stress

Elastic Response with Random G

- General form of elastic constitutive rate equation

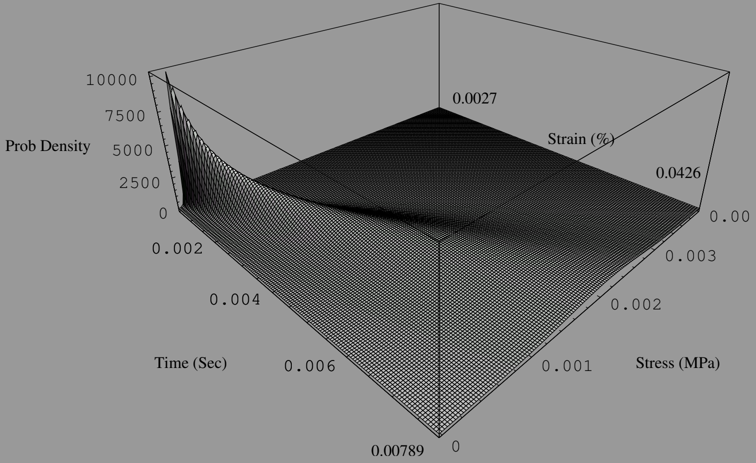
$$\begin{aligned}\frac{d\sigma_{12}}{dt} &= 2G\frac{d\epsilon_{12}}{dt}\\ &= \eta(G,\epsilon_{12};t)\end{aligned}$$

- Advection and diffusion coefficients of FPK equation

$$N_{(1)} = 2\frac{d\epsilon_{12}}{dt} < G >$$

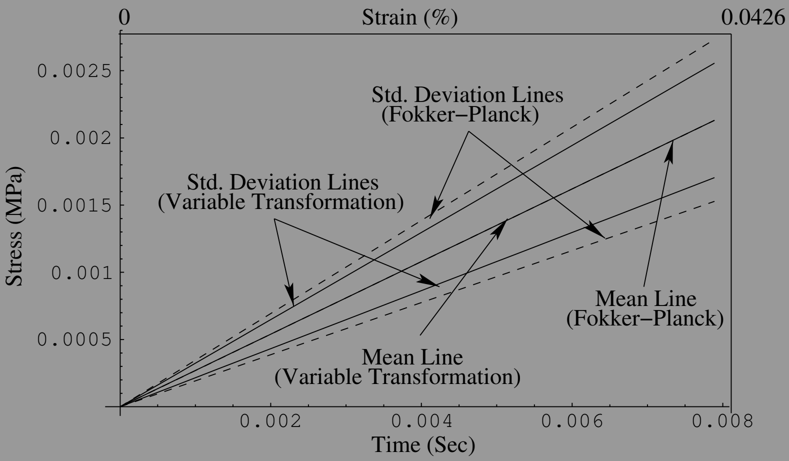
$$N_{(2)} = 4t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var[G]$$

Elastic Response with Random G



$< G >$ = 2.5 MPa; Std. Deviation$[G]$ = 0.5 MPa

Verification – Variable Transformation Method



Drucker-Prager Linear Hardening with Random G

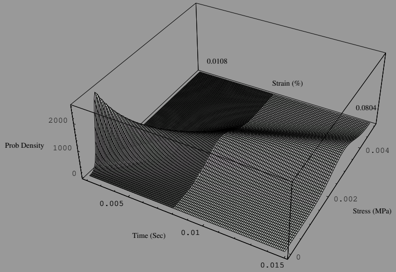
$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, K, \alpha, \alpha', \epsilon_{12}; t)$$

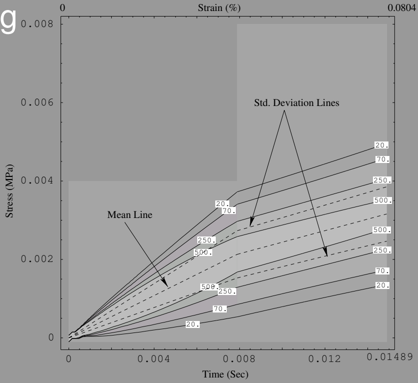
Advection and diffusion coefficients of FPK equation

$$N_{(1)} = \frac{d\epsilon_{12}}{dt}\left\langle 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right\rangle$$

$$N_{(2)} = t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var\left[2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right]$$

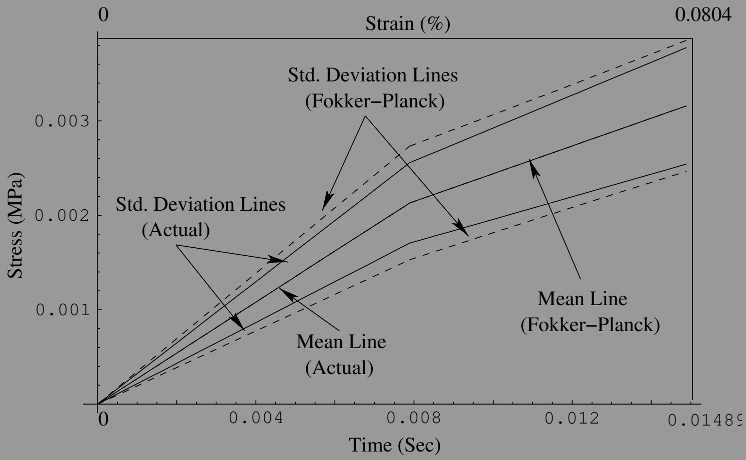
Drucker-Prager Linear Hardening with Random *G*





- Approximation of I.C.
- Smooth transition between el. & el.-pl.
- Symmetry in probability distribution

Verification of D–P E–P Response - Monte Carlo



Modified Cam Clay Constitutive Model

$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, M, e_0, p_0, \lambda, \kappa, \epsilon_{12}; t)$$

$$\eta = \left[2G - \frac{\left(36\frac{G^2}{M^4}\right)\sigma_{12}^2}{\frac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\frac{G}{M^4}\right)\sigma_{12}^2 + \frac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)} \right]$$

Advection and diffusion coefficients of FPK equation

$$N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov\left[\frac{\partial \eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau)\right]$$

$$N_{(2)}^{(i)} = \int_0^t d\tau cov\left[\eta^{(i)}(t); \eta^{(i)}(t-\tau)\right]$$

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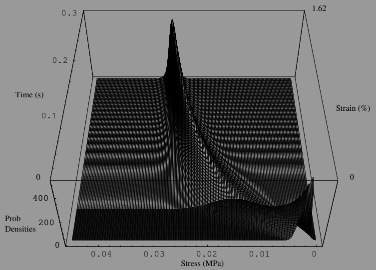
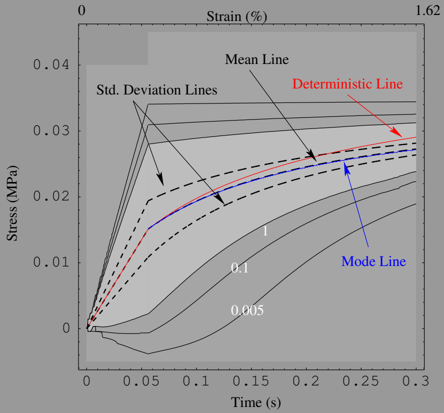
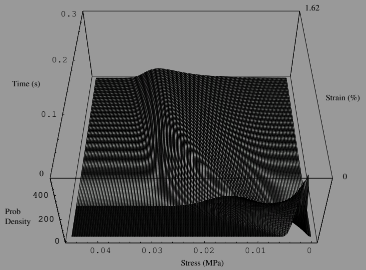
Low OCR Cam Clay with Random *G*

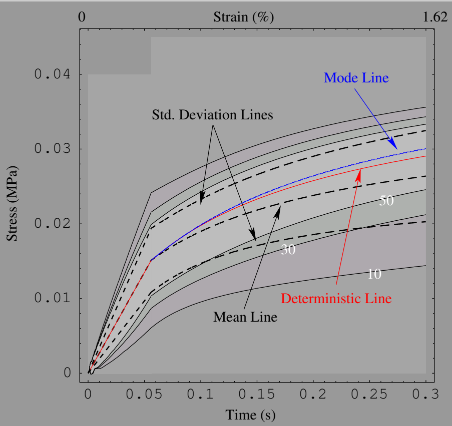




- Approximation of I.C.
- Non-symmetry in probability distribution!
- Response at critical state fairly certain but different than deterministic

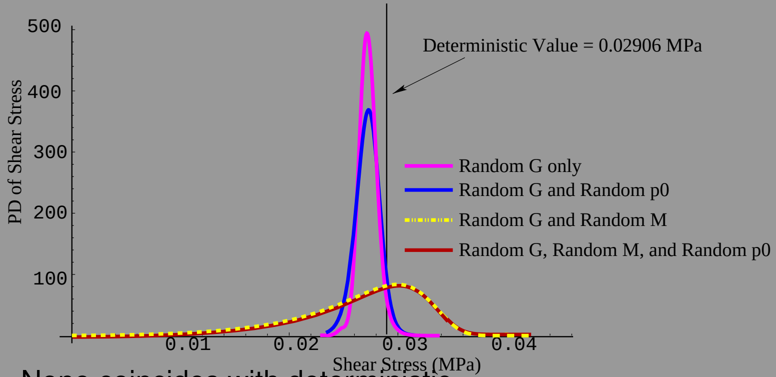
Low OCR Cam Clay with Random G, M and p_0





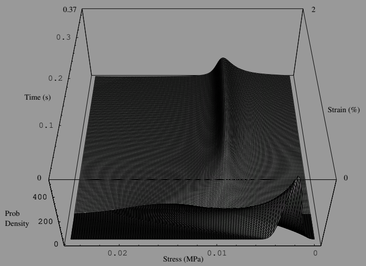
- Non-symmetry in probability distribution
- Difference between mean, mode and deterministic responses
- Divergence at critical state because M is uncertain

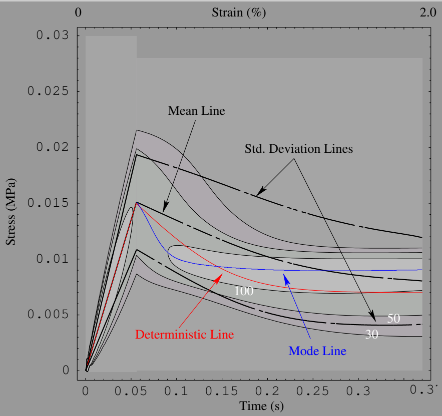
Comparison of Low OCR Cam Clay at ϵ = 1.62 %



- None coincides with deterministic
- Some cases are very uncertain while some are fairly certain
- Either on safe or unsafe side

High OCR Cam Clay with Random *G* and *M*





- Very uncertain transition between el. & el.-pl.
- **Differences** between mean, mode, and deterministic responses
- Divergence at critical state, *M* is uncertain

Conclusions

- A new approach to account for uncertainties in elastic–plastic material simulation
- Methodology, which results in a FPK equation, overcomes the drawbacks of *Monte Carlo Method* and *Perturbation Technique*
- Advantage of FPK equation is evident as it transforms the original non–linear stochastic ODE to a linear deterministic PDE
- Developed methodology is capable of providing complete probabilistic description (PDF) of the solution
- Development is general in nature and applicable to any incremental elastic–plastic material model