Uncertain Elasto–Plasticity

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Outline

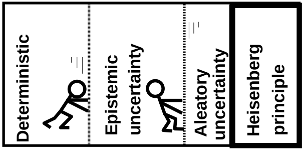
Outline

History

- Probabilistic fish counting
- Williams' DEM simulations, differential displacement vortexes
- SFEM round table
- Kavvas' probabilistic hydrology

Types of Uncertainties

- Epistemic uncertainty - due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools are not well developed
  - trade-off with aleatory uncertainty
- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools



Ergodicity

- Exchange ensemble averages for time averages
- Is soil elasto-plasticity ergodic?
  - Can soil elastic–plastic statistical properties be obtained by temporal averaging?
  - Will soil elastic–plastic statistical properties "renew" at each occurrence?
  - Are soil elastic–plastic statistical properties statistically independent?
- Claim in literature that structural nonlinear behavior is non–ergodic while earthquake characteristics are (?!)
- However, earthquake characteristics is representing mechanics (fault slip) on a different scale...

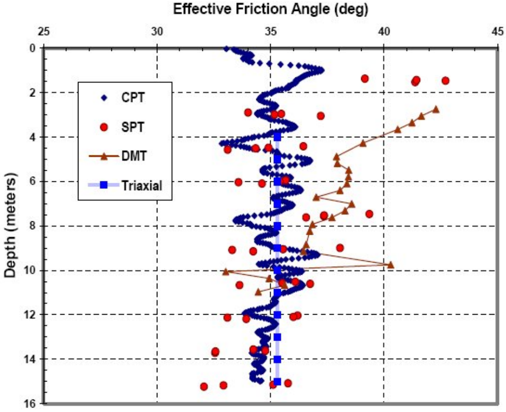
Historical Overview

- Brownian motion, Langevin equation → PDF governed by simple diffusion Eq. (Einstein 1905)
- With external forces → Fokker-Planck-Kolmogorov (FPK) for the PDF (Kolmogorov 1941)
- Approach for random forcing → relationship between the autocorrelation function and spectral density function (Wiener 1930)
- Approach for random coefficient → Functional integration approach (Hopf 1952), Averaged equation approach (Bharrucha-Reid 1968), Numerical approaches, Monte Carlo method

Outline

Material Behavior Inherently Uncertainties

- Spatial variability
- Point-wise uncertainty, testing error, transformation error



(Mayne et al. (2000))

Soil Uncertainties and Quantification

- Natural variability of soil deposit (Fenton 1999)
  - Function of soil formation process
- Testing error (Stokoe et al. 2004)
  - Imperfection of instruments
  - Error in methods to register quantities
- Transformation error (Phoon and Kulhawy 1999)
  - Correlation by empirical data fitting (e.g. CPT data → friction angle etc.)

Probabilistic material (Soil Site) Characterization

- Ideal: complete probabilistic site characterization
- Large (physically large but not statistically) amount of data
  - Site specific mean and coefficient of variation (COV)
  - Covariance structure from similar sites (e.g. Fenton 1999)
- Moderate amount of data → Bayesian updating (e.g. Phoon and Kulhawy 1999, Baecher and Christian 2003)
- Minimal data: general guidelines for typical sites and test methods (Phoon and Kulhawy (1999))
  - COVs and covariance structures of inherent variability
  - COVs of testing errors and transformation uncertainties.

Recent State-of-the-Art

- Governing equation
  - Dynamic problems → $M\ddot{u} + C\dot{u} + Ku = \phi$
  - Static problems → $Ku = \phi$
- Existing solution methods
  - **Random r.h.s** (external force random)
    - FPK equation approach
    - Use of fragility curves with deterministic FEM (DFEM)
  - **Random l.h.s** (material properties random)
    - Monte Carlo approach with DFEM → CPU expensive
    - Perturbation method → a linearized expansion! Error increases as a function of COV
    - Spectral method → developed for elastic materials so far
- New developments for elasto–plastic applications

Outline

Uncertainty Propagation through Constitutive Eq.

- Incremental el–pl constitutive equation $\frac{d\sigma_{ij}}{dt} = D_{ijkl}\frac{d\epsilon_{kl}}{dt}$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{for elastic} \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} & \text{for elastic–plastic} \end{cases}$$

Previous Work

- Linear algebraic or differential equations → Analytical solution:
  - Variable Transf. Method (Montgomery and Runger 2003)
  - Cumulant Expansion Method (Gardiner 2004)
- Nonlinear differential equations (elasto-plastic/viscoelastic-viscoplastic):
  - Monte Carlo Simulation (Schueller 1997, De Lima et al 2001, Mellah et al. 2000, Griffiths et al. 2005...)
    → accurate, very costly
  - Perturbation Method (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 1997)
    → first and second order Taylor series expansion about mean - limited to problems with small C.O.V. and inherits "closure problem"

Problem Statement

- Incremental 3D elastic-plastic stress–strain:

$$\frac{d\sigma_{ij}}{dt} = \left\{ D^{el}_{ijkl} - \frac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} \right\} \frac{d\epsilon_{kl}}{dt}$$

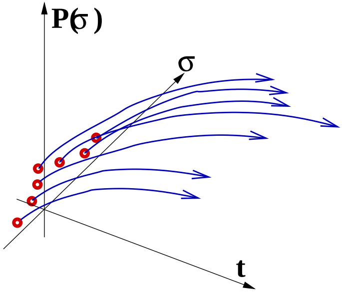
- Focus on 1D $\rightarrow$ a nonlinear ODE with random coefficient (material) and random forcing (ϵ)

$$\begin{aligned} \frac{d\sigma(x,t)}{dt} &= \beta(\sigma(x,t), D^{el}(x), q(x), r(x); x, t) \frac{d\epsilon(x,t)}{dt} \\ &= \eta(\sigma, D^{el}, q, r, \epsilon; x, t) \end{aligned}$$

with initial condition $\sigma(0) = \sigma_0$

Evolution of the Density $\rho(\sigma, t)$

- From each initial point in σ-space a trajectory starts out describing the corresponding solution of the stochastic process
- Movement of a cloud of initial points described by density $\rho(\sigma, 0)$ in σ-space, is governed by the constitutive equation,



Stochastic Continuity (Liouville) Equation

- phase density ρ of $\sigma(x,t)$ varies in time according to a continuity Liouville equation (Kubo 1963):

$$\frac{\partial \rho(\sigma(x,t),t)}{\partial t} = -\frac{\partial \eta(\sigma(x,t), D^{el}(x), q(x), r(x), \epsilon(x,t))}{\partial \sigma} \rho[\sigma(x,t),t]$$

- with initial conditions $\rho(\sigma, 0) = \delta(\sigma - \sigma_0)$

Ensemble Average form of Liouville Equation

Continuity equation written in ensemble average form (eg. cumulant expansion method (Kavvas and Karakas 1996)):

$$
\begin{aligned}
\frac{\partial \langle \rho(\sigma(x_t,t),t)\rangle}{\partial t} &= -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t,t),D^{el}(x_t),q(x_t),r(x_t),\epsilon(x_t,t))\right\rangle\right.\right. \\
&+ \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(x_t,t),D^{el}(x_t),q(x_t),r(x_t),\epsilon(x_t,t))}{\partial\sigma};\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau),D^{el}(x_{t-\tau}),q(x_{t-\tau}),r(x_{t-\tau}),\epsilon(x_{t-\tau},t-\tau)\right]\right\}\langle\rho(\sigma(x_t,t),t)\rangle\right] \\
&+ \frac{\partial^2}{\partial\sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(x_t,t),D^{el}(x_t),q(x_t),r(x_t),\epsilon(x_t,t));\right.\right.\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau),D^{el}(x_{t-\tau}),q(x_{t-\tau}),r(x_{t-\tau}),\epsilon(x_{t-\tau},t-\tau))\right]\right\}\langle\rho(\sigma(x_t,t),t)\rangle\right]
\end{aligned}
$$

Eulerian–Lagrangian FPK Equation

van Kampen's Lemma (van Kampen 1976) $\rightarrow < \rho(\sigma, t) >= P(\sigma, t)$, ensemble average of phase density is the probability density;

$$
\begin{aligned}
\frac{\partial P(\sigma(x_t,t),t)}{\partial t} &= -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))\right\rangle\right.\right. \\
&+ \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))}{\partial \sigma};\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau},t-\tau)\right]\right\} P(\sigma(x_t,t),t)\right] \\
&+ \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t));\right.\right.\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau},t-\tau))\right]\right\} P(\sigma(x_t,t),t)\right]
\end{aligned}
$$

Eulerian–Lagrangian Format

- Real-space location (Lagrangian) x_t is known but pull-back to Eulerian location $x_{t-\tau}$ is unknown
- Can be related using strain rate $\dot{\epsilon}$ $(= d\epsilon/dt)$

$$d\epsilon = \dot{\epsilon}\tau = \frac{x_t - x_{t-\tau}}{x_t}; \quad \text{or,} \quad x_{t-\tau} = (1 - \dot{\epsilon}\tau)x_t$$

E–L FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right]$$

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- It is deterministic equation in probability density space
- It is linear PDE in probability density space → Simplifies the numerical solution process

B. Jeremić, K. Sett, and M. L. Kavvas, "Probabilistic Elasto–Plasticity: Formulation in 1–D", *Acta Geotechnica*, Vol. 2, No. 3, 2007, In press (published online in the *Online First* section)

Template Solution of FPK Equation

- FPK diffusion–advection equation is applicable to any material model → only the coefficients $N_{(1)}$ and $N_{(2)}$ are different for different material models

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition
  - Deterministic → Dirac delta function → $P(\sigma, 0) = \delta(\sigma)$
  - Random → Any given distribution
- Boundary condition: Reflecting BC → conserves probability mass $\zeta(\sigma,t)|_{At\ Boundaries} = 0$
- Finite Differences used for solution (among many others)

K. Sett, B. Jeremić and M.L. Kavvas, "The Role of Nonlinear Hardening/Softening in Probabilistic Elasto–Plasticity", *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol. 31, No. 7, pp. 953-975, 2007

Application of FPK equation to Material Models

- FPK equation is applicable to any incremental elastic–plastic material model
- Solution in terms of PDF, not a single value of stress
- Influence of initial condition on the PDF of stress
- Mean stress yielding or
- Probabilistic yielding

Outline

Elastic Response with Random G

- General form of elastic constitutive rate equation

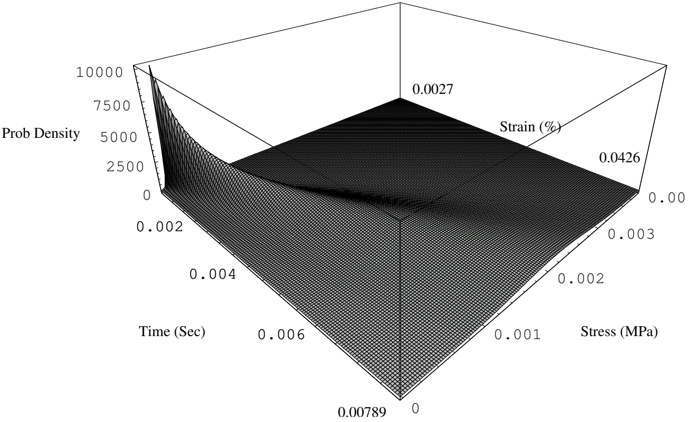
$$\frac{d\sigma_{12}}{dt} = 2G\frac{d\epsilon_{12}}{dt} = \eta(G, \epsilon_{12}; t)$$

- Advection and diffusion coefficients of FPK equation

$$N_{(1)} = 2\frac{d\epsilon_{12}}{dt} < G >$$

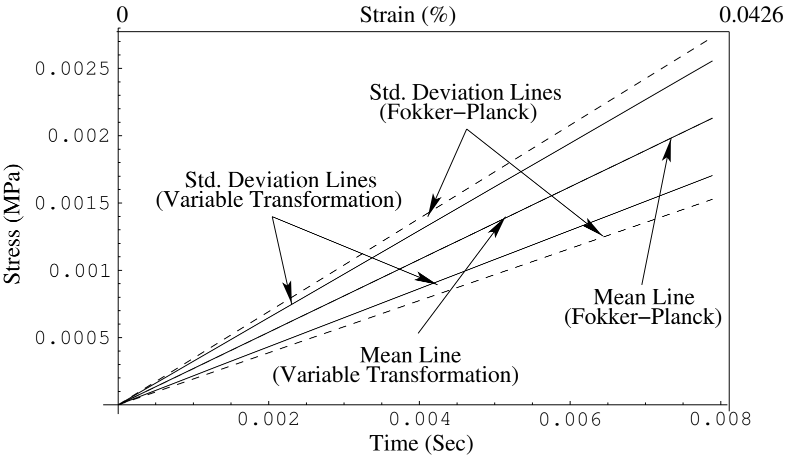
$$N_{(2)} = 4t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var[G]$$

Elastic Response with Random *G*



$< G >$ = 2.5 MPa; Std. Deviation[G] = 0.5 MPa

Verification – Variable Transformation Method



Drucker-Prager Linear Hardening with Random G

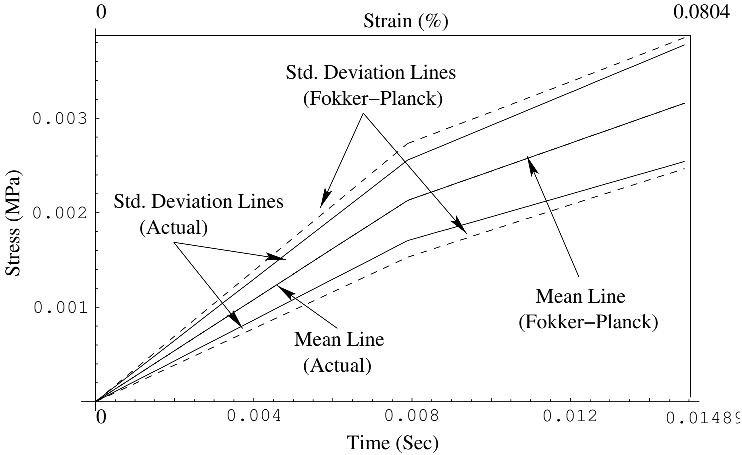
$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, K, \alpha, \alpha', \epsilon_{12}; t)$$

Advection and diffusion coefficients of FPK equation

$$N_{(1)} = \frac{d\epsilon_{12}}{dt}\left\langle 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1\alpha'}\right\rangle$$

$$N_{(2)} = t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var\left[2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1\alpha'}\right]$$

Verification of D–P E–P Response - Monte Carlo



Modified Cam Clay Constitutive Model

$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, M, e_0, p_0, \lambda, \kappa, \epsilon_{12}; t)$$

$$\eta = \left[2G - \frac{\left(36\frac{G^2}{M^4}\right)\sigma_{12}^2}{\frac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\frac{G}{M^4}\right)\sigma_{12}^2 + \frac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)}\right]$$

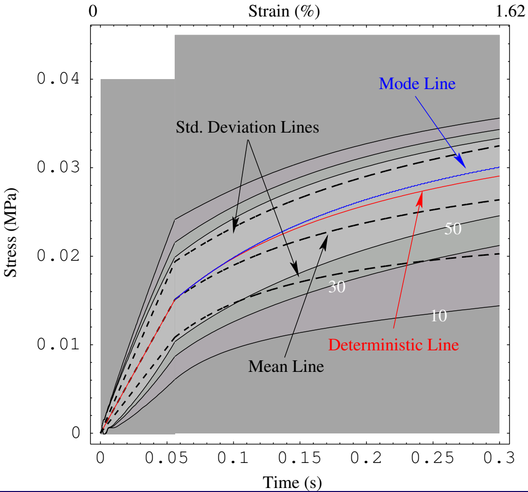
Advection and diffusion coefficients of FPK equation

$$N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov\left[\frac{\partial\eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau)\right]$$

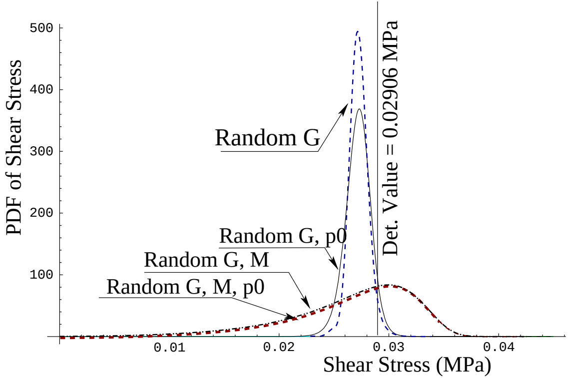
$$N_{(2)}^{(i)} = \int_0^t d\tau cov\left[\eta^{(i)}(t); \eta^{(i)}(t-\tau)\right]$$

Low OCR Cam Clay with Random G, M and p_0

- Non-symmetry in probability distribution
- Difference between mean, mode and deterministic
- Divergence at critical state because M is uncertain



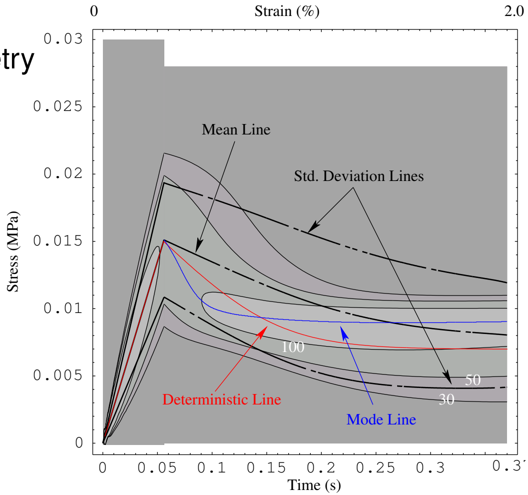
Comparison of Low OCR Cam Clay at ϵ = 1.62 %



- None coincides with deterministic
- Some very uncertain, some very certain
- Either on safe or unsafe side

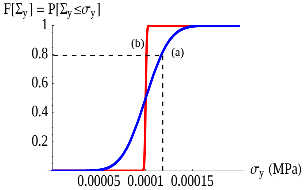
High OCR Cam Clay with Random *G* and *M*

- Large non-symmetry in probability distribution
- Significant differences in mean, mode, and deterministic
- Divergence at critical state, *M* is uncertain



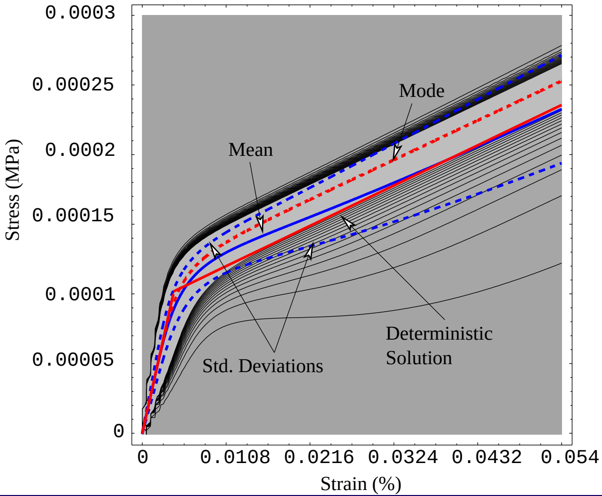
Probabilistic Yielding

- Weighted elastic and elastic–plastic Solution
  $$\partial P(\sigma,t)/\partial t = -\partial\left(N^{w}_{(1)} P(\sigma,t) - \partial\left(N^{w}_{(2)} P(\sigma,t)\right)\partial\sigma\right)/\partial\sigma$$
- Weighted advection and diffusion coefficients are then
  $$N^{w}_{(1,2)}(\sigma) = (1 - P[\Sigma_y \le \sigma])N^{el}_{(1)} + P[\Sigma_y \le \sigma]N^{el-pl}_{(1)}$$
- Cumulative Probability Density function (CDF) of the yield function



B. Jeremić and K. Sett. On Probabilistic Yielding of Materials. in review in *Communications in Numerical Methods in Engineering*, 2007.

Transformation of a Bi–Linear (von Mises) Response



Outline

Governing Equations & Discretization Scheme

- Governing equations of mechanics:

$$A\sigma = \phi(t); \quad Bu = \epsilon; \quad \sigma = D\epsilon$$

- Discretization (spatial and stochastic) schemes
  - Input random field material properties (D) $\longrightarrow$ Karhunen–Loève (KL) expansion, optimal expansion, error minimizing property
  - Unknown solution random field (u) $\longrightarrow$ Polynomial Chaos (PC) expansion
  - Deterministic spatial differential operators (A & B) $\longrightarrow$ Regular shape function method with Galerkin scheme

Spectral Stochastic Elastic–Plastic FEM

- Minimizing norm of error of finite representation using Galerkin technique (Ghanem and Spanos 2003):

$$\sum_{n=1}^{N} K_{mn} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} C_{ijk} K'_{mnk} = \langle F_m \psi_i[\{\xi_r\}] \rangle$$

$$K_{mn} = \int_D B_n D B_m dV \qquad K'_{mnk} = \int_D B_n \sqrt{\lambda}_k h_k B_m dV$$

$$C_{ijk} = \langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \rangle \qquad F_m = \int_D \phi N_m dV$$

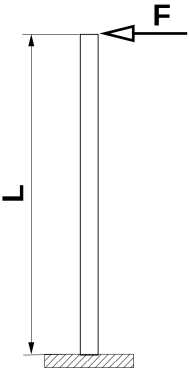
Inside SEPFEM

- Explicit stochastic elastic–plastic finite element computations
- FPK probabilistic constitutive integration at Gauss integration points
- Increase in (stochastic) dimensions (KL and PC) of the problem (parallelism)
- Development of the probabilistic elastic–plastic stiffness tensor

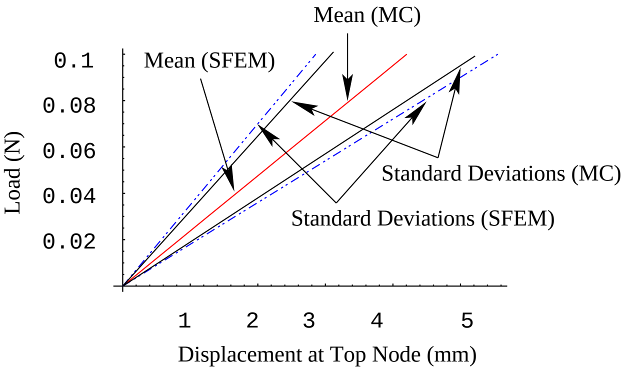
Outline

1–D Static Pushover Test Example

- Linear elastic model:
  $< G >= 2.5$ kPa,
  $Var[G] = 0.15\ \mathrm{kPa}^2$,
  correlation length for $G = 0.3$ m.
- Elastic–plastic material model,
  von Mises, linear hardening,
  $< G >= 2.5$ kPa,
  $Var[G] = 0.15\ \mathrm{kPa}^2$,
  correlation length for $G = 0.3$ m,
  $C_u = 5$ kPa,
  $C_u' = 2$ kPa.

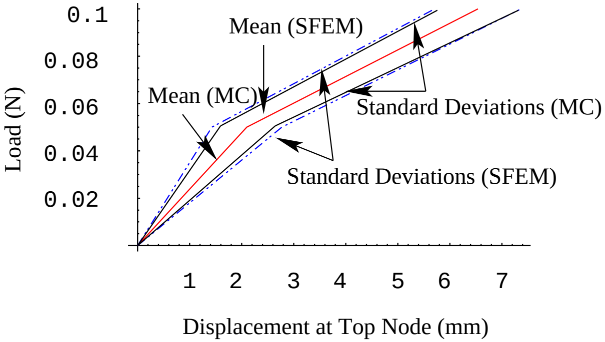


Linear Elastic FEM Verification



Mean and standard deviations of displacement at the top node, linear elastic material model, KL-dimension=2, order of PC=2.

SEPFEM verification



Mean and standard deviations of displacement at the top node, von Mises elastic-plastic linear hardening material model, KL-dimension=2, order of PC=2.

Outline

Applications

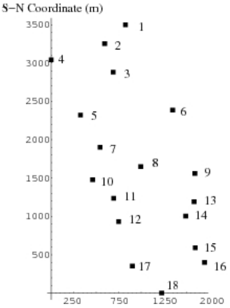
- Stochastic elastic–plastic simulations of soils and structures
- Probabilistic inverse problems
- Geotechnical site characterization design
- Optimal material design

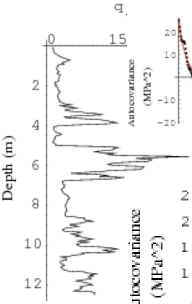
Seismic Wave Propagation through Stochastic Soil

- Soil as 12.5 m deep 1–D soil column (von Mises Material)
  - Properties (including testing uncertainty) obtained through random field modeling of CPT q_T
    $\langle q_T \rangle = 4.99\ MPa$; $Var[q_T] = 25.67\ MPa^2$;
    Cor. Length $[q_T] = 0.61\ m$; Testing Error $= 2.78\ MPa^2$
- q_T was transformed to obtain G: $G/(1-\nu) = 2.9 q_T$
  - Assumed transformation uncertainty = 5%
    $\langle G \rangle = 11.57 MPa$; $Var[G] = 142.32 MPa^2$
    Cor. Length $[G] = 0.61 m$
- Input motions: modified 1938 Imperial Valley

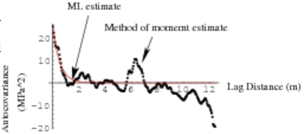
Random Field Parameters from Site Data

- Maximum likelihood estimates

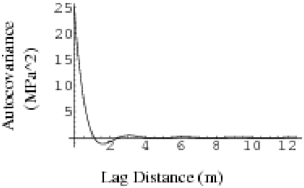




Typical CPT q_T

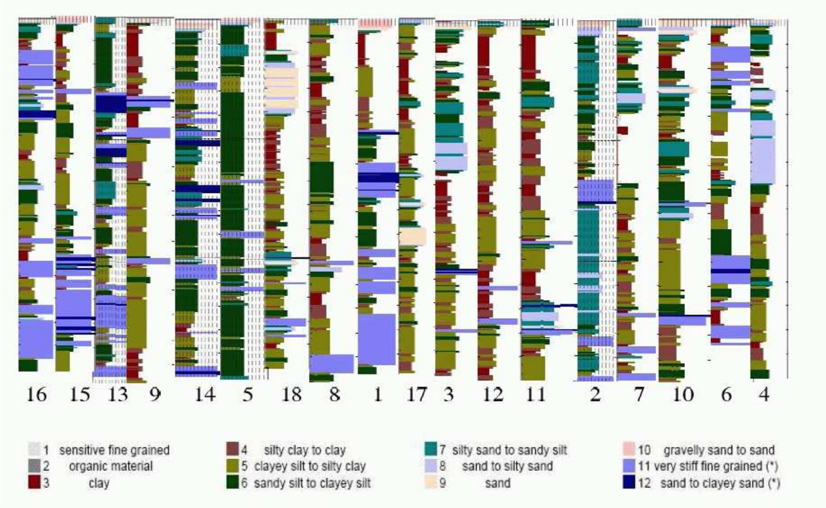


Finite Scale

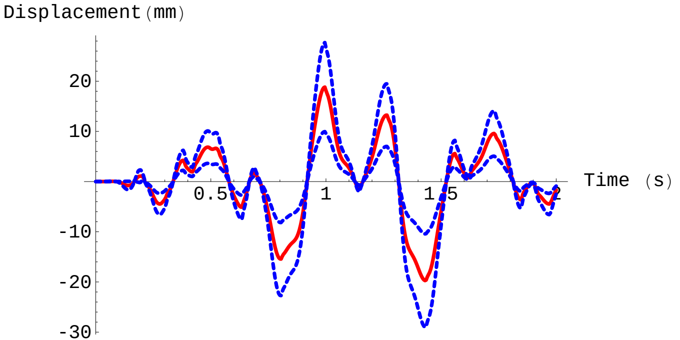


Fractal

Variable (Uniform) CPT Site Data



Seismic Wave Propagation through Stochastic Soil



Mean± Standard Deviation

Summary

- Developed a second-order (mean and variance) exact, method to account for probabilistic elastic–plastic material simulation
- In combination with SSFEM, PEP methodology allows simulations of both point–wise and spatial uncertainty of elastic–plastic materials
- Consistent modeling of spatial and point-wise uncertainties in material properties for static and dynamic behaviors of solids and structures
- Probably numerous applications

Thank You