On Probabilistic Yielding of (Geo–)Materials

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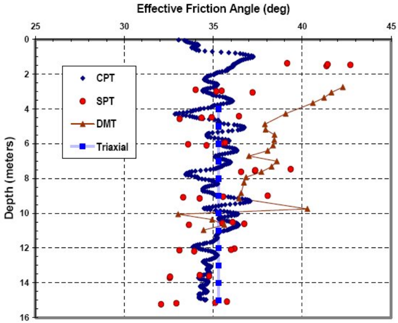
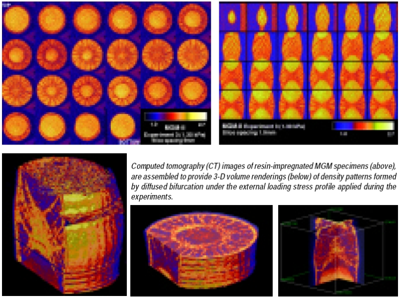
Outline

Uncertain (Geo–)materials



Mayne et al. (2000)



Sture et al. (1998)

Background

- Brownian motion, Langevin equation (Einstein 1905)
- Random forcing (Fokker-Planck, Kolmogorov 1941)
- Random coefficient (Hopf 1952), (Bharrucha-Reid 1968), Monte Carlo method
- Epistemic uncertainty, lack of knowledge
- Aleatory uncertainty, inherent variation
- Methods for treating epistemic uncertainty not well developed, trade-off with aleatory uncertainty
- Ergodicity of geomaterials !(?)
- Prof. Einav question

Outline

Uncertainty Propagation through Constitutive Eq.

- Incremental el–pl constitutive equation $\Delta\sigma_{ij} = D_{ijkl}\Delta\epsilon_{kl}$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{for elastic} \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} & \text{for elastic–plastic} \end{cases}$$

Eulerian–Lagrangian FPK Equation

Liouville continuity equation (Kubo 1963); ensemble average form (Kavvas and Karakas 1996); van Kampen's Lemma (van Kampen 1976)

$$
\begin{aligned}
\frac{\partial P(\sigma(x_t,t),t)}{\partial t} &= -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))\right\rangle\right.\right. \\
&+ \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))}{\partial \sigma};\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau},t-\tau)\right]\right\} P(\sigma(x_t,t),t)\right] \\
&+ \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t));\right.\right.\right. \\
&\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau},t-\tau))\right]\right\} P(\sigma(x_t,t),t)\right]
\end{aligned}
$$

Compact Form of Eulerian–Lagrangian FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right]$$

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- It is deterministic equation in probability density space
- It is linear PDE in probability density space → simplifies the numerical solution process

B. Jeremić, K. Sett, and M. L. Kavvas, "Probabilistic Elasto–Plasticity: Formulation in 1–D", *Acta Geotechnica*, Vol. 2, No. 3, 2007

Template Solution of E–L FPK Equation

- FPK diffusion–advection equation is applicable to any material model $\rightarrow$ only the coefficients $N_{(1)}$ and $N_{(2)}$ are different for different material models

$$\frac{\partial P(\sigma, t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)} P(\sigma, t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)} P(\sigma, t)\right\}\right]$$

- Initial condition random or deterministic
- Solution for both
  - probabilistic elastic (PEL) and
  - probabilistic elastic–plastic (PELPL) problems

K. Sett, B. Jeremić and M.L. Kavvas, "The Role of Nonlinear Hardening/Softening in Probabilistic Elasto–Plasticity", *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol. 31, No. 7, pp. 953-975, 2007

Outline

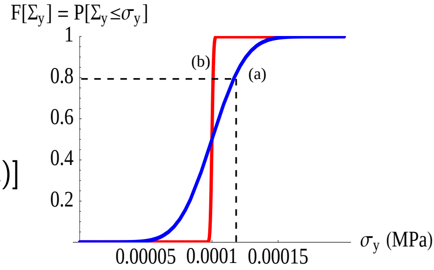
Probabilistic Yielding: Weighted Coefficients

- Probabilistic yielding: take both PEL and PELPL into account at the same time with some (un–)certainty
- Weighted elastic and elastic–plastic Solution
  $\partial P(\sigma,t)/\partial t = -\partial \left(N^{w}_{(1)} P(\sigma,t) - \partial \left(N^{w}_{(2)} P(\sigma,t) \right)/\partial\sigma \right)/\partial\sigma$
- Weighted advection and diffusion coefficients are then
  $N^{w}_{(1,2)}(\sigma) = (1 - P[\Sigma_y \le \sigma]) N^{el}_{(1)} + P[\Sigma_y \le \sigma] N^{el-pl}_{(1)}$
- Similar to European Pricing Option in financial simulations (Black–Scholes options pricing model '73, Nobel prize for Economics '97)

B. Jeremić and K. Sett. On Probabilistic Yielding of Materials. in print in *Communications in Numerical Methods in Engineering*, 2008.

Probability of Yielding

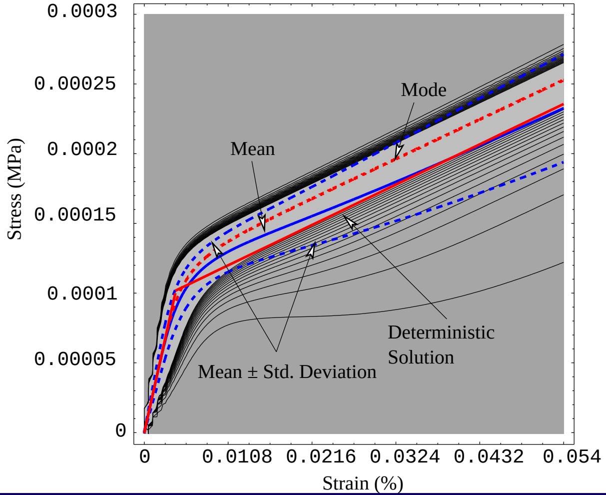
- Probabilistic von Mises
- Probability of yielding at $\sigma = 0.0012\ \mathrm{MPa}$ $P[\Sigma_y \leq (\sigma = 0.0012\ \mathrm{MPa})] = 0.8$
- Equivalent advection and diffusion coefficients are



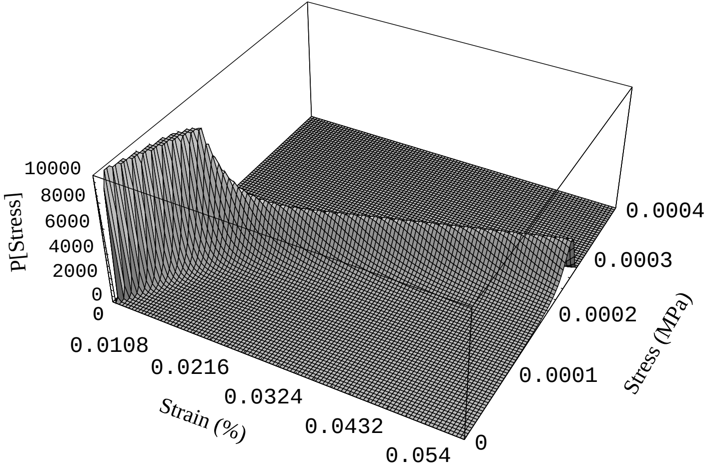
$$N_{(1)}^{eq}|_{\sigma=0.0012\ MPa} = (1-0.8)N_{(1)}^{el} + 0.8N_{(1)}^{ep}$$

$$N_{(2)}^{eq}|_{\sigma=0.0012\ MPa} = (1-0.8)N_{(2)}^{el} + 0.8N_{(2)}^{ep}$$

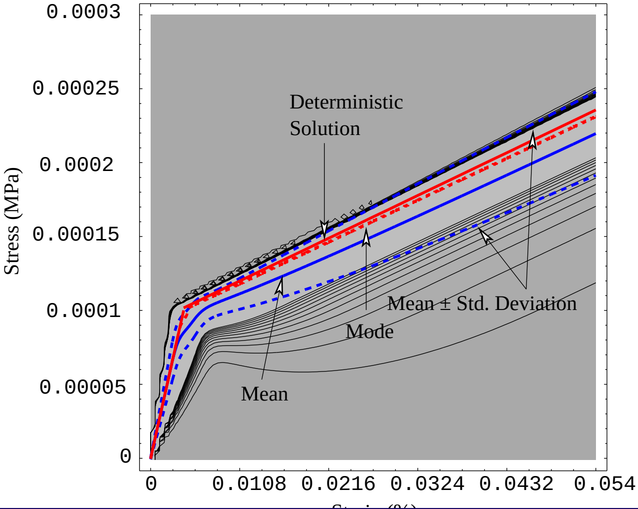
Bi–Linear von Mises Response



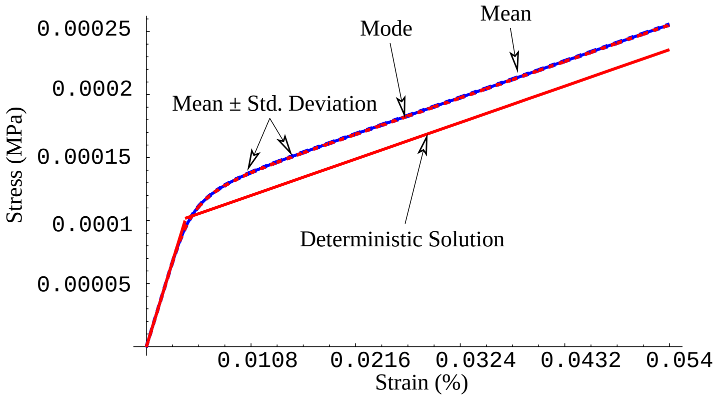
Bi–Linear von Mises Response



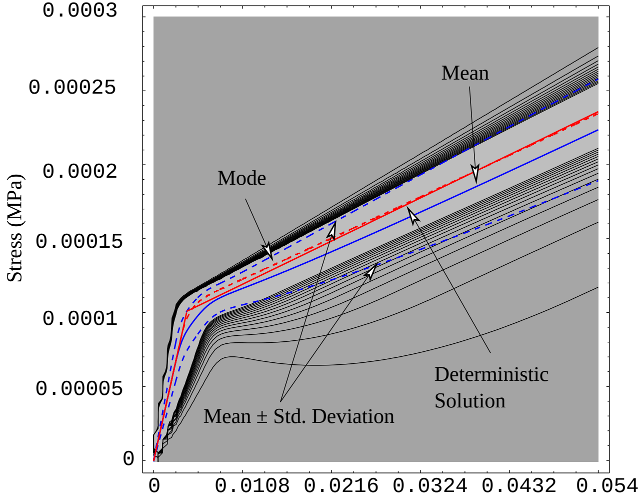
von Mises, Certain c_u, Uncertain G



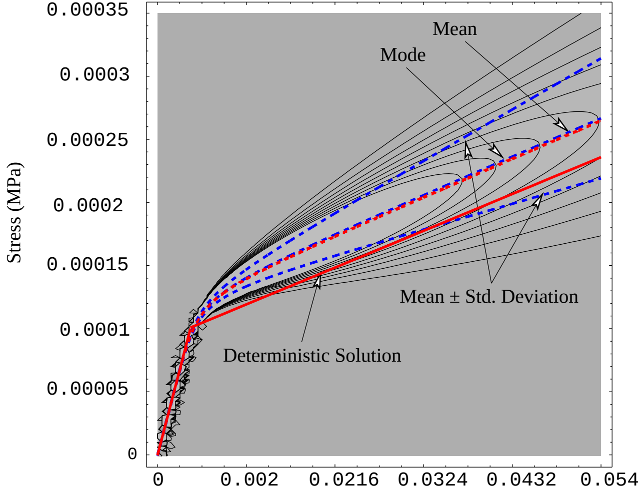
von Mises, Uncertain c_u, Certain G



Drucker Prager: Uncertain G, Certain ϕ



Drucker Prager: Certain *G*, Uncertain ϕ



Summary

- Second-order (mean and variance) exact, method to account for probabilistic elastic–plastic material simulation
- Probabilities of material yielding, probably govern underlying mechanics of elasto–plasticity
- Probabilities of material yielding (spatial distributions) also probably govern underlying mechanics of failure, localization of deformation...
- Probably numerous applications


Thank You