Elastic–Plastic Behavior of Geomaterials: Modeling and Simulation Issues

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Outline

Motivation

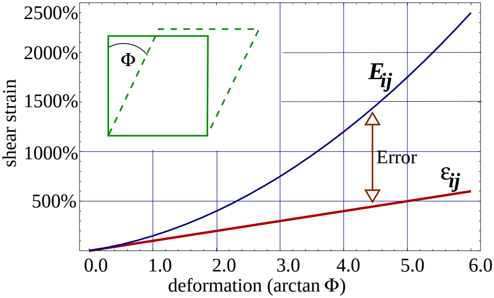
- Use well developed theory of elasto–plasticity for modeling and simulating geomatarials
- Issues at the *constitutive* and the *finite element* levels
- Verification and Validation is very important
- There is no limit to what problems one can address (can numerically simulate)

Loading Process

- Stages
- Increments
- Iterations

Outline

Small Deformation



$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{i,k}u_{k,j}\right) \quad ; \quad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Elasticity

- Hyperelasticity, $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$ (where W is the strain energy function per unit volume)
- Hypoelasticity, direct modeling of nonlinear elastic deformation, not thermodynamically consistent
- Linear and nonlinear elastic models

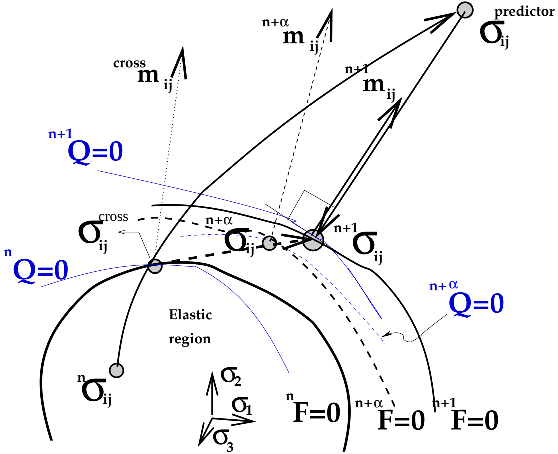
Incremental Elasto–Plasticity

- Additive decomposition of strain $\Delta\epsilon_{ij} = \Delta\epsilon^{e}_{ij} + \Delta\epsilon^{p}_{ij}$
- Elastic relationship (generalized Hooke's law)
  $\Delta\sigma_{ij} = E_{ijkl}\Delta\epsilon^{e}_{kl}$
- (non) Associated plastic flow rule
  $\Delta\epsilon^{p}_{ij} = \Delta\lambda\, \partial Q/\partial\sigma_{ij} = \Delta\lambda\, m_{ij}(\sigma_{ij}, q_{*})$
- Hardening/softening (isotropic/anisotropic) law
  $\Delta q_{*} = \Delta\lambda\, h_{*}(\tau_{ij}, q_{*})$

Karush–Kuhn–Tucker Conditions

- Yield function $F(\sigma_{ij}, q_*) \leq 0$
- Plastic consistency parameter $\Delta\lambda \geq 0$
- loading – unloading condition $F\ \Delta\lambda = 0$

Midpoint Integration Algorithm

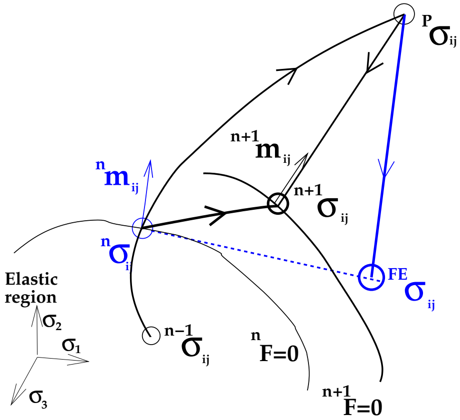


Midpoint Integration Algorithm

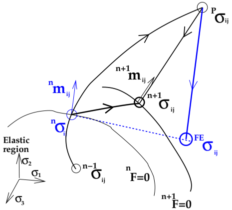
- Rarely used (even if for $\alpha = 0.5$ it is second order accurate)
- Explicit algorithm ($\alpha = 0.0$)
- Implicit algorithm ($\alpha = 1.0$)

Outline

Explicit Integration Algorithm



Explicit Integration Algorithm

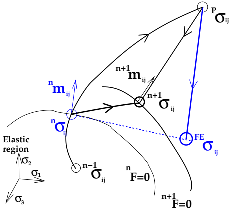


Increments

$$\Delta\sigma_{mn} = E_{mnpq}\ \Delta\epsilon_{pq} - E_{mnpq}\ \frac{{}^{n}n_{rs}\ E_{rstu}\ \Delta\epsilon_{tu}}{{}^{n}n_{ab}\ E_{abcd}\ {}^{n}m_{cd} - \xi_A h_A}\ {}^{n}m_{pq}$$

$$\Delta q_A = \left(\frac{{}^{n}n_{mn}\ E_{mnpq}\ \Delta\epsilon_{pq}}{{}^{cros}n_{mn}\ E_{mnpq}\ {}^{cros}m_{pq} - \xi_A h_A} \right) h_A$$

Explicit Integration Algorithm

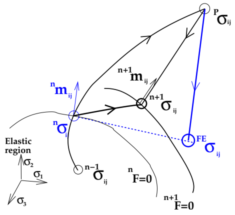


Tangent stiffness

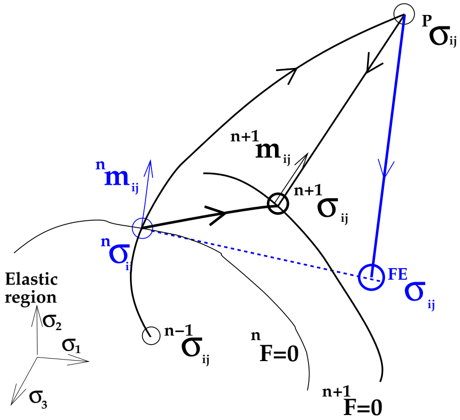
$$ {}^{cont}E^{ep}_{pqmn} = E_{pqmn} - \frac{E_{pqkl}\,{}^{n}m_{kl}\,{}^{n}n_{ij}\,E_{ijmn}}{{}^{n}n_{ot}\,E_{otrs}\,{}^{n}m_{rs} - {}^{n}\xi_A\, h_A} $$

Explicit Integration Algorithm

- Relatively simple (first derivatives)
- Fast (single step)
- Inaccurate (accumulates error)
- Popular (most/all commercial codes)
- Works well with global explicit algorithm

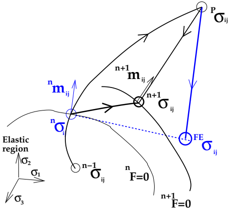


Implicit Integration Algorithm

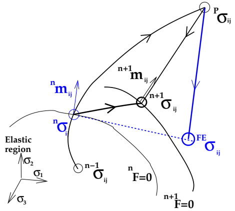


Implicit Integration Algorithm

- Also based on elastic predictor – plastic corrector
  $${}^{n+1}\sigma_{ij} = {}^{pred}\sigma_{ij} - \Delta\lambda\; E_{ijkl}\; {}^{n+1}m_{kl}$$
- Tensor of residuals used in iterations
  $$r_{ij} = \sigma_{ij} - \left({}^{pred}\sigma_{ij} - \Delta\lambda\; E_{ijkl}\; m_{kl}\right)$$
- Iterative increments
  $$\mathrm{d}(\Delta\lambda) = \left({}^{old}f - \mathbf{n}^T\;\mathbb{C}\;{}^{old}\mathbf{r}\right) / \left(\mathbf{n}^T\;\mathbb{C}\;\mathbf{M}\right)$$
  $$\left\{\begin{array}{c} \mathrm{d}\sigma_{mn} \\ \mathrm{d}q_B \end{array}\right\} = -\mathbb{C}\left({}^{old}\mathbf{r} + \mathrm{d}(\Delta\lambda)\mathbf{m}\right)$$
  with $\mathbf{n} = \left\{\begin{array}{c} n_{mn} \\ \xi_B \end{array}\right\}$, $\mathbf{m} = \left\{\begin{array}{c} E_{ijkl}m_{kl} \\ -h_A \end{array}\right\}$, ${}^{old}\mathbf{r} = \left\{\begin{array}{c} {}^{old}\sigma_{ij} \\ {}^{old}r_A \end{array}\right\}$



Implicit Integration Algorithm



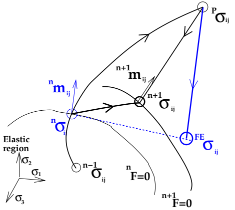
- Super-matrix $\mathbb{C}$ has different formats depending on a number and type of internal variables

$$\mathbb{C} = \left[\; I^{s}_{ijmn} + \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} \; \right]^{-1}$$

$$\mathbb{C} = \begin{bmatrix} I^{s}_{ijmn} + \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} & \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial q} \\ -\Delta\lambda \frac{\partial h}{\partial \sigma_{ij}} & 1 - \Delta\lambda \frac{\partial h}{\partial q} \end{bmatrix}^{-1}$$

$$\mathbb{C} =$$
$$\begin{bmatrix} I^{s}_{ijmn} + \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} & \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial z_{mn}} & \Delta\lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \alpha_{mn}} \\ -\Delta\lambda \frac{\partial h^{z}}{\partial \sigma_{ij}} & I^{s}_{ijmn} - \Delta\lambda \frac{\partial h^{z}}{\partial z_{mn}} & -\Delta\lambda \frac{\partial h^{z}}{\partial \alpha_{ij}} \\ -\Delta\lambda \frac{\partial h^{\alpha}_{mn}}{\partial \sigma_{ij}} & -\Delta\lambda \frac{\partial h^{\alpha}_{mn}}{\partial z_{mn}} & I^{s}_{ijmn} - \Delta\lambda \frac{\partial h^{\alpha}_{mn}}{\partial \alpha_{ij}} \end{bmatrix}^{-1}$$

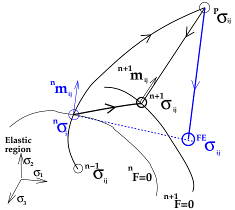
Implicit Integration Algorithm



Consistent (algorithmic) stiffness

$$\left\{ \begin{array}{c} d\sigma_{ij} \\ dq_A \end{array} \right\} = \left\{ \mathbb{C} - \frac{\mathbb{C}\mathbf{m}\mathbf{n}^T\mathbb{C}}{\mathbf{n}^T\,\mathbb{C}\,\mathbf{m}} \right\} \left\{ \begin{array}{c} E_{ijmn}\, d\epsilon_{mn}^{pred} \\ 0 \end{array} \right\}$$

Implicit Integration Algorithm



- Relatively complicated (first and second derivatives, inverse)
- Relatively slow (but improves global Newton iterations)
- Accurate (consistency condition satisfied at the end, within tolerance)
- Popular for research
- Unpopular in commercial codes (except simple material models)
- Designed to work with global Newton algorithm

Outline

Principle of Virtual Displacements

$$\int_V \sigma_{ij} \, \delta\epsilon_{ij} \, dV = \int_V \left(f_i^B - \rho \ddot{u}_i \right) \, \delta u_i \, dV + \int_S f_i^S \, \delta u_i \, dS$$

Discretization

$$u \approx \hat{u}_a = H_I \bar{u}_{Ia}$$

$$
\begin{aligned}
\epsilon_{ab} \approx \hat{e}_{ab} &= \frac{1}{2}\left(\hat{u}_{a,b} + \hat{u}_{b,a}\right) = \\
&= \frac{1}{2}\left(\left(H_I\, \bar{u}_{Ia}\right)_{,b} + \left(H_I\, \bar{u}_{Ib}\right)_{,a}\right) = \\
&= \frac{1}{2}\left(\left(H_{I,b}\, \bar{u}_{Ia}\right) + \left(H_{I,a}\, \bar{u}_{Ib}\right)\right)
\end{aligned}
$$

FEM Equations

$$\bigcup_{(m)} {}^{(m)}M_{IacJ}\ \ddot{\bar{u}}_{Jc} + \bigcup_{(m)} {}^{(m)}K_{IacJ}\ \bar{u}_{Jc} = \bigcup_{m} {}^{(m)}F^{B}_{Ia} + \bigcup_{m} {}^{(m)}F^{S}_{Ia}$$

$$ {}^{(m)}M_{IacJ} = \int_{V^m} H_J\ \delta_{ac}\ \rho\ H_I\ dV^m \quad ; \quad {}^{(m)}F^{B}_{Ia} = \int_{V^m} f^{B}_{a}\ H_I\ dV^m$$

$$ {}^{(m)}K_{IacJ} = \int_{V^m} H_{I,b}\ E_{abcd}\ H_{J,d}\ dV^m \quad ; \quad {}^{(m)}F^{S}_{Ia} = \int_{S^m} f^{S}_{a}\ H_I\ dS^m$$

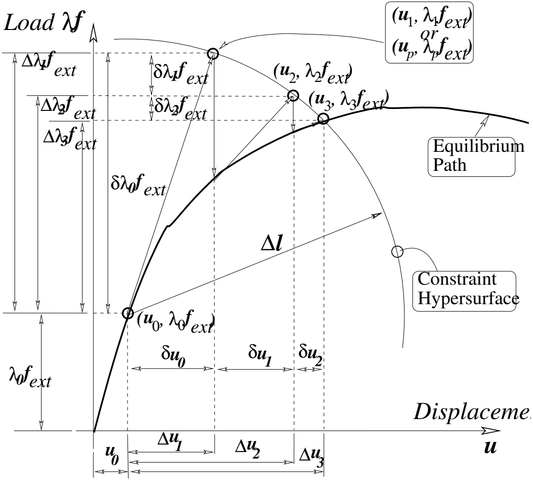
Outline

Residual Force Equation in Statics

$$r_i(u_j, \lambda) = f_i^{int}(u_j) - \lambda f_i^{ext} = 0$$

- $f_i^{int}(u_j)$ are the internal forces which are functions of the displacements u_j,
- f_i^{ext} is a *fixed external loading vector*
- λ is a *load–level parameter*
- Proportional loading

Advancing the Solution



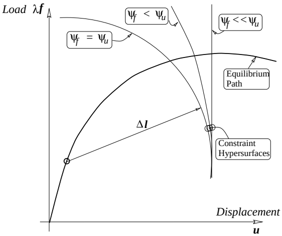
Hyper–spherical Constraint

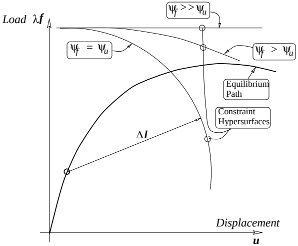
$$s = \int ds \quad \text{where} \quad ds = \sqrt{\frac{\psi_u^2}{u_{ref}^2} du_i S_{ij} du_j + d\lambda^2 \psi_f^2}$$

or, in incremental form:

$$a = (\Delta s)^2 - (\Delta l)^2 = \left(\frac{\psi_u^2}{u_{ref}^2} \Delta u_i S_{ij} \Delta u_i + \Delta\lambda^2 \psi_f^2 \right) - (\Delta l)^2$$

Specializations





- Coefficients ψ_u and ψ_f may not be simultaneously zero
- If $S_{ij} = I_{ij}$ and $u_{ref} = 1 \rightarrow$ arclength method
- If $S_{ij} = K_{ij}^t$ and $\psi_f \equiv 0 \rightarrow$ external work constraint
- If $\psi_u \equiv 0$ and $\psi_f \equiv 1 \rightarrow$ load control
- If $\psi_u \equiv 1$, $\psi_f \equiv 0$ and $S_{ij} = I_{ij} \rightarrow$ generalized displacement control

Following the Equilibrium Path in Statics

- Family of Newton methods (full, initial stress, modified...)
- Traversing equilibrium path in positive sense (positive external work criterion; angle criterion)
- Accuracy control
- Numerical stability
- Automatic increments
- Convergence criteria (absolute, relative, force and/or displacement and/or energy based)

Transient Integration Algorithms

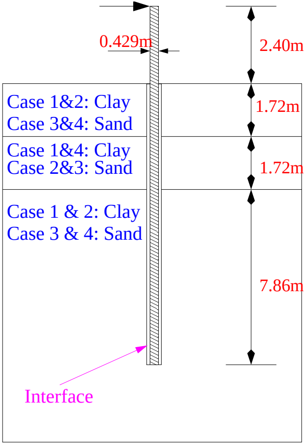
- Finite differences, simple, but inacurate
- Wilson $\theta = 1.37$, too much numerical damping
- Newmark, controllable numerical damping, period elongation
  $\gamma \geq 1/2, \quad \beta = 1/4(\gamma + 1/2)^2$
- Hilber–Hughes–Taylor, extension of Newmark with better damping
  $-1/3 \leq \alpha \leq 0, \quad \gamma = 1/2(1-2\alpha), \quad \beta = 1/4(1-\alpha)^2$

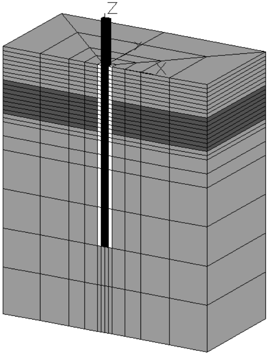
Dynamic Analysis

- Stability (artificial introduction of higher frequencies by discretization process)
- Accuracy, conservation of energy and period
- Time step choice (the shorter the better, unless too many (artificial) high frequencies are present).
- multiple DOF type systems (u-p-U, structural elements...)

Outline

Single Pile in Layered Soils: Model

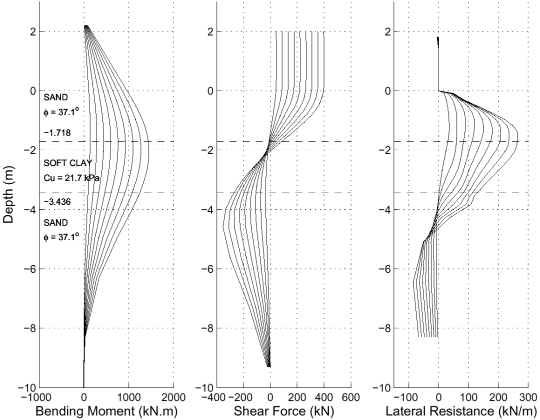




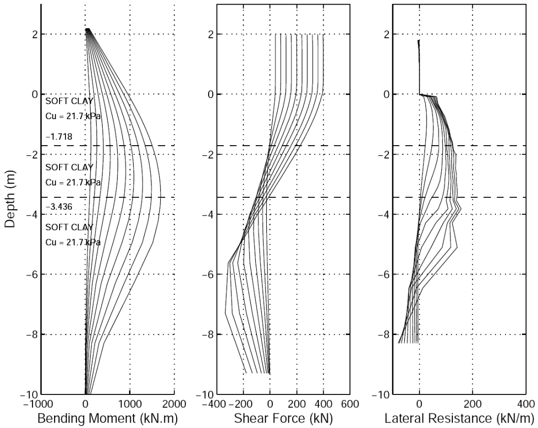
Available Data for Prototype

- Sand:
  - friction angle ϕ of 37.1^o,
  - Shear modulus at a depth of 13.7 m of 8960 kPa (E_o = 17400 kPa),
  - Poisson ratio of 0.35
  - Unit weight of 14.50 kN/m^3.
  - Dilation angle 0^o
- Clay (made up)
  - Shear strength 21.7 kPa
  - Young's modulus 11000 kPa
  - Poisson ratio 0.45
  - Unit weight 13.7 $\mathrm{kN/m^3}$

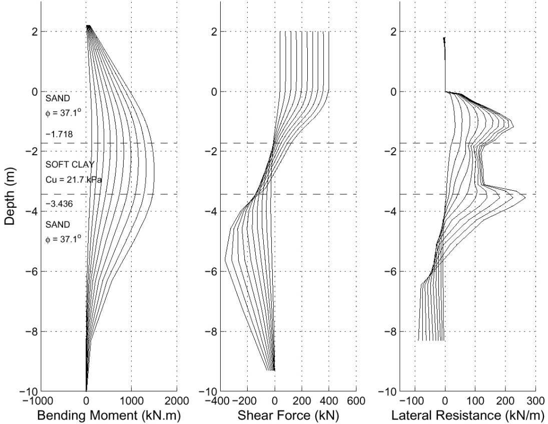
Single Pile in Sand: M, Q, p



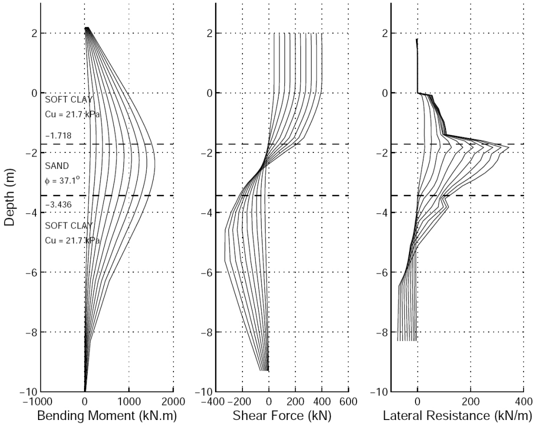
Single Pile in Clay: M, Q, p



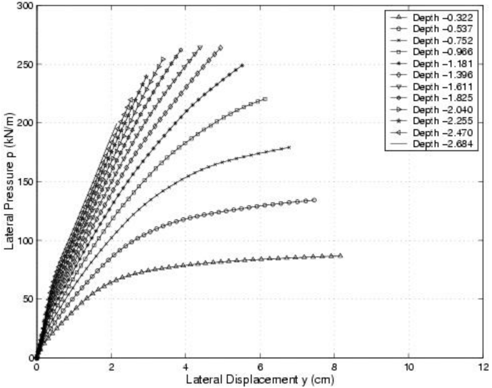
Single Pile in Sand with Clay Layer: M, Q, p

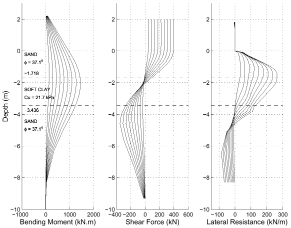


Single Pile in Clay with Sand Layer: M, Q, p



Single Pile in Sand: $p - y$ Response





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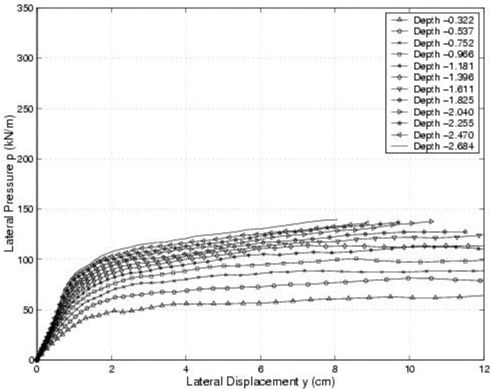
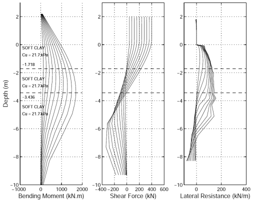
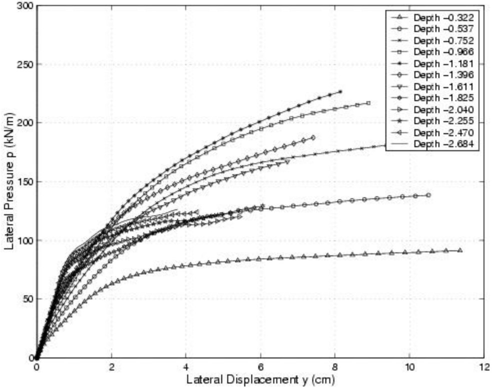
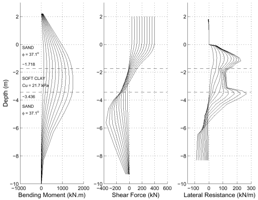


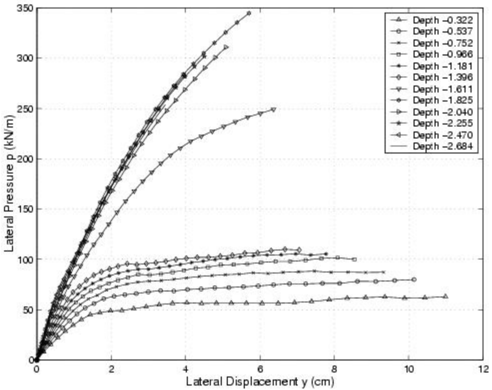


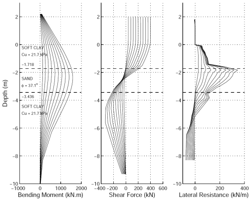
Single Pile in Sand with Clay Layer: $p-y$ Response



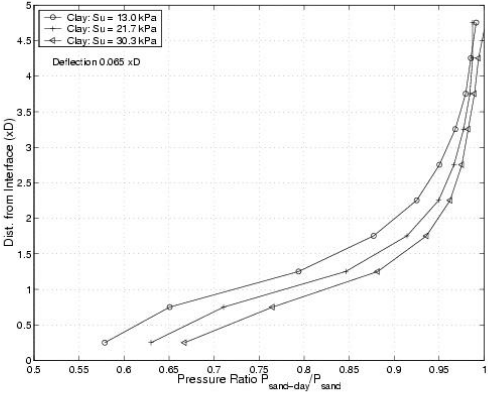


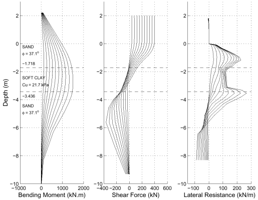
Single Pile in Clay with Sand Layer: $p - y$ Response





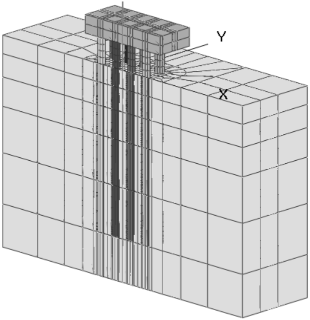
$p - y$ Pressure Ratio Reduction for Layered Soils

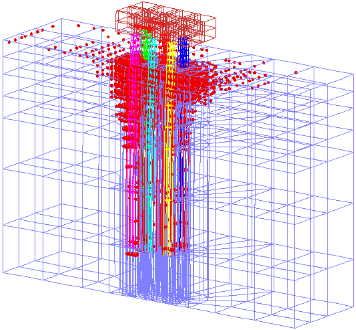




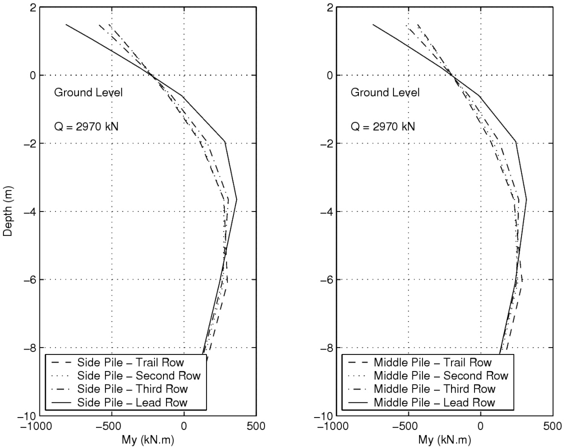
Outline

Pile Group Simulations

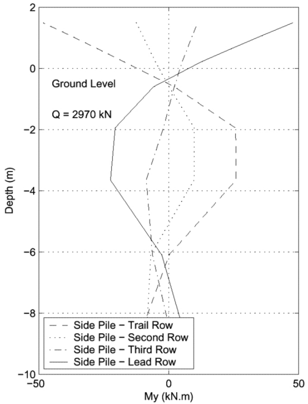




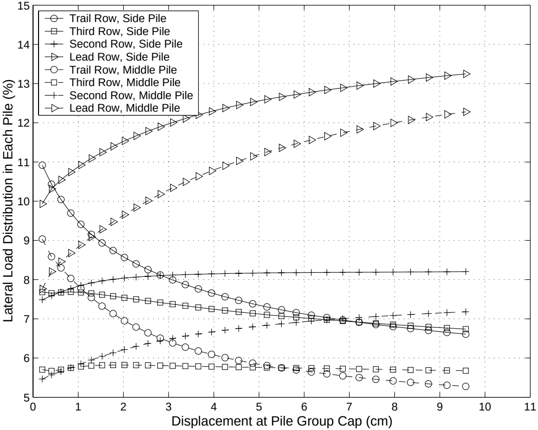
Bending Moments



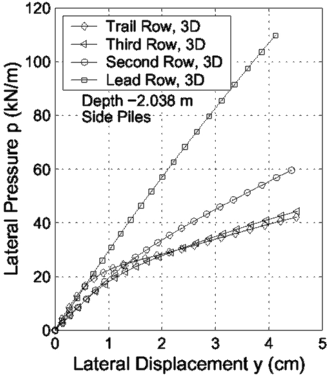
Out of Plane Effects

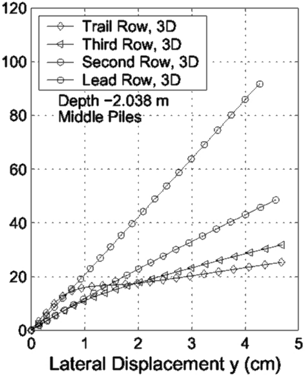


Load Distribution per Pile



Piles Interaction at -2.0m ($p - y$)





Summary

- Importance of consistent formulation, material modeling and implementation
- Verified, validate models and simulations tools used for prediction of behavior
- Program and examples available in public domain (Author's web site)