Elastic–Plastic Behavior of Geomaterials: Modeling and Simulation Issues

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Outline

Before We Start

Constitutive Level
- Small Deformation Elasto–Plasticity
- Explicit and Implicit Constitutive Integrations

Finite Element Level
- Formulation
- Statics and Dynamics

Examples
- Piles
- Pile Groups

Summary
Before We Start

### Motivation

- Use well developed theory of elasto–plasticity for modeling and simulating geomaterials
- Issues at the *constitutive* and the *finite element* levels
- Verification and Validation is very important
- There is no limit to what problems one can address (can numerically simulate)
Loading Process

- Stages
- Increments
- Iterations
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Summary
Small Deformation

\[ E_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + u_{i,k} u_{k,j} \right) ; \quad \varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \]
Small Deformation Elasto–Plasticity

### Elasticity

- Hyperelasticity, $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$ (where $W$ is the strain energy function per unit volume)

- Hypoelasticity, direct modeling of nonlinear elastic deformation, not thermodynamically consistent

- Linear and nonlinear elastic models
Incremental Elasto–Plasticity

- Additive decomposition of strain $\Delta \epsilon_{ij} = \Delta \epsilon_{ij}^e + \Delta \epsilon_{ij}^p$

- Elastic relationship (generalized Hooke’s law)
  $\Delta \sigma_{ij} = E_{ijkl} \Delta \epsilon_{kl}^e$

- (non) Associated plastic flow rule
  $\Delta \epsilon_{ij}^p = \Delta \lambda \frac{\partial Q}{\partial \sigma_{ij}} = \Delta \lambda \ m_{ij}(\sigma_{ij}, q_*)$

- Hardening/softening (isotropic/anisotropic) law
  $\Delta q_* = \Delta \lambda \ h_*(\tau_{ij}, q_*)$
Karush–Kuhn–Tucker Conditions

- Yield function $F(\sigma_{ij}, q_\star) \leq 0$
- Plastic consistency parameter $\Delta \lambda \geq 0$
- Loading – unloading condition $F \Delta \lambda = 0$
Midpoint Integration Algorithm
Midpoint Integration Algorithm

- Rarely used (even if for $\alpha = 0.5$ it is second order accurate)
- Explicit algorithm ($\alpha = 0.0$)
- Implicit algorithm ($\alpha = 1.0$)
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Explicit Integration Algorithm
Explicit Integration Algorithm

Increments

\[ \Delta \sigma_{mn} = E_{mn pq} \Delta \epsilon_{pq} - E_{mn pq} \frac{n_{rs} E_{rstu} \Delta \epsilon_{tu}}{n_{ab} E_{abcd} n_{cd} - \xi_A h_A} n_{mpq} \]

\[ \Delta q_A = \left( \frac{n_{mn} E_{mn pq} \Delta \epsilon_{pq}}{c_{ros} n_{mn} E_{mn pq} c_{ros} m_{pq} - \xi_A h_A} \right) h_A \]
Explicit Integration Algorithm

Tangent stiffness

\[
\begin{align*}
\text{cont} E^{ep}_{pqmn} &= E_{pqmn} - \frac{E_{pqkl}^{n} m_{kl}^{n} n_{ij} E_{ijmn}}{n_{ot} E_{ots} n_{rs}^{n} m_{rs} - \zeta_{A} h_{A}}
\end{align*}
\]
Explicit Integration Algorithm

- Relatively simple (first derivatives)
- Fast (single step)
- Inaccurate (accumulates error)
- Popular (most/all commercial codes)
- Works well with global explicit algorithm
Implicit Integration Algorithm
Implicit Integration Algorithm

- Also based on elastic predictor – plastic corrector
  \[ n+1\sigma_{ij} = \text{pred}\sigma_{ij} - \Delta \lambda \ E_{ijkl} \ n+1m_{kl} \]

- Tensor of residuals used in iterations
  \[ r_{ij} = \sigma_{ij} - (\text{pred}\sigma_{ij} - \Delta \lambda \ E_{ijkl} \ m_{kl}) \]

- Iterative increments
  \[ d(\Delta \lambda) = (\text{old}f - \ n^T \ C \ \text{old}r) / (\ n^T \ C \ M) \]
  \[ \begin{cases} 
  d\sigma_{mn} \\
  dq_B \end{cases} = -C \ (\text{old}r + d(\Delta \lambda)m) \]
  with \[ n = \begin{bmatrix} n_{mn} \\
  \xi_B \end{bmatrix}, \ m = \begin{bmatrix} E_{ijkl}m_{kl} \\
  -h_A \end{bmatrix}, \ \text{old}r = \begin{bmatrix} \text{old}\sigma_{ij} \\
  \text{old}r_A \end{bmatrix} \]
Explicit and Implicit Constitutive Integrations

Implicit Integration Algorithm

- Super-matrix $C$ has different formats depending on a number and type of internal variables

$$C = \left[ I^S_{ijmn} + \Delta \lambda E_{ijkl} \frac{\partial m_{kl}}{\partial \sigma_{mn}} \right]^{-1}$$

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Implicit Integration Algorithm

Consistent (algorithmic) stiffness

\[
\begin{bmatrix}
\frac{d\sigma_{ij}}{d\sigma}
\end{bmatrix}
= 
\begin{bmatrix}
C - \frac{Cmn^T C}{n^T C m}
\end{bmatrix}
\begin{bmatrix}
E_{ijmn} d\epsilon_{mn}^{pred}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}
= 
\begin{bmatrix}
0
\end{bmatrix}
\]
Implicit Integration Algorithm

- Relatively complicated (first and second derivatives, inverse)
- Relatively slow (but improves global Newton iterations)
- Accurate (consistency condition satisfied at the end, within tolerance)
- Popular for research
- Unpopular in commercial codes (except simple material models)
- Designed to work with global Newton algorithm
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Principle of Virtual Displacements

\[ \int_V \sigma_{ij} \delta \epsilon_{ij} \, dV = \int_V \left( f^B_i - \rho \ddot{u}_i \right) \delta u_i \, dV + \int_S f^S_i \delta u_i \, dS \]
Discretization

\[ u \approx \hat{u}_a = H_l \ddot{u}_{la} \]

\[ \varepsilon_{ab} \approx \hat{\varepsilon}_{ab} = \frac{1}{2} \left( \hat{u}_{a,b} + \hat{u}_{b,a} \right) = \frac{1}{2} \left( (H_l \ddot{u}_{la})_b + (H_l \ddot{u}_{lb})_a \right) = \frac{1}{2} \left( (H_{l,b} \ddot{u}_{la}) + (H_{l,a} \ddot{u}_{lb}) \right) \]
FEM Equations

\[ \bigcup_{(m)} (l_{acJ}) \ddot{u}_{Jc} + \bigcup_{(m)} (k_{acJ}) \ddot{u}_{Jc} = \bigcup_{m} (f_B^I) + \bigcup_{m} (f_S^I) \]

\[ (l_{acJ}) = \int_{V_m} H_j \delta_{ac} \rho H_l \, dV^m ; \quad (f_B^I) = \int_{V_m} f_{a}^{B} H_l \, dV^m \]

\[ (k_{acJ}) = \int_{V_m} H_{l,b} E_{abcd} H_{J,d} \, dV^m ; \quad (f_S^I) = \int_{S_m} f_{a}^{S} H_l \, dS^m \]
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Residual Force Equation in Statics

\[ r_i(u_j, \lambda) = f_i^{\text{int}}(u_j) - \lambda f_i^{\text{ext}} = 0 \]

- \( f_i^{\text{int}}(u_j) \) are the internal forces which are functions of the displacements \( u_j \),
- \( f_i^{\text{ext}} \) is a fixed external loading vector
- \( \lambda \) is a load–level parameter
- Proportional loading
Advancing the Solution

Load $\lambda f$

$\lambda f_{\text{ext}}$

$\lambda f_{\text{ext}}$

$\lambda f_{\text{ext}}$

Displacement $u$

$u_0$

$\Delta u_1$

$\Delta u_2$

$\Delta u_3$

Equilibrium Path

Constraint Hypersurface

$(u_1, \lambda_{f_{\text{ext}}})$

$(u_2, \lambda_{f_{\text{ext}}})$

$(u_3, \lambda_{f_{\text{ext}}})$

$(u_p, \lambda_{f_{\text{ext}}})$
Hyper–spherical Constraint

\[ s = \int ds \quad \text{where} \quad ds = \sqrt{\frac{\psi_u^2}{u_{\text{ref}}^2} u_i S_{ij} u_j + d\lambda^2 \psi_f^2} \]

or, in incremental form:

\[ a = (\Delta s)^2 - (\Delta l)^2 = \left( \frac{\psi_u^2}{u_{\text{ref}}^2} \Delta u_i S_{ij} \Delta u_i + \Delta \lambda^2 \psi_f^2 \right) - (\Delta l)^2 \]
Specializations

- Coefficients $\psi_u$ and $\psi_f$ may not be simultaneously zero
- If $S_{ij} = I_{ij}$ and $u_{ref} = 1 \rightarrow$ arclength method
- If $S_{ij} = K_{ij}^t$ and $\psi_f \equiv 0 \rightarrow$ external work constraint
- If $\psi_u \equiv 0$ and $\psi_f \equiv 1 \rightarrow$ load control
- If $\psi_u \equiv 1$, $\psi_f \equiv 0$ and $S_{ij} = I_{ij} \rightarrow$ generalized displacement control
Following the Equilibrium Path in Statics

- Family of Newton methods (full, initial stress, modified...)
- Traversing equilibrium path in positive sense (positive external work criterion; angle criterion)
- Accuracy control
- Numerical stability
- Automatic increments
- Convergence criteria (absolute, relative, force and/or displacement and/or energy based)
Transient Integration Algorithms

- Finite differences, simple, but inaccurate

- Wilson $\theta = 1.37$, too much numerical damping

- Newmark, controllable numerical damping, period elongation
  $$\gamma \geq 1/2, \quad \beta = 1/4(\gamma + 1/2)^2$$

- Hilber–Hughes–Taylor, extension of Newmark with better damping
  $$-1/3 \leq \alpha \leq 0, \quad \gamma = 1/2(1 - 2\alpha), \quad \beta = 1/4(1 - \alpha)^2$$
Dynamic Analysis

- Stability (artificial introduction of higher frequencies by discretization process)
- Accuracy, conservation of energy and period
- Time step choice (the shorter the better, unless too many (artificial) high frequencies are present).
- multiple DOF type systems (u-p-U, structural elements...)
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Single Pile in Layered Soils: Model

Case 1 & 2: Clay
Case 3 & 4: Sand
Case 1 & 4: Clay
Case 2 & 3: Sand

Interface

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On the topic of pile foundations, some key data points for the prototype are available:

- **Sand:**
  - Friction angle $\phi$ of 37.1°,
  - Shear modulus at a depth of 13.7 m of 8960 kPa ($E_o = 17400$ kPa),
  - Poisson ratio of 0.35
  - Unit weight of 14.50 kN/m$^3$.
  - Dilation angle 0°

- **Clay (made up):**
  - Shear strength 21.7 kPa
  - Young’s modulus 11000 kPa
  - Poisson ratio 0.45
  - Unit weight 13.7 kN/m$^3$
Single Pile in Sand: $M, Q, p$
Single Pile in Clay: M, Q, p
Single Pile in Sand with Clay Layer: M, Q, p
Single Pile in Clay with Sand Layer: M, Q, p
Single Pile in Sand: $p - y$ Response
Single Pile in Clay: $p - y$ Response
Single Pile in Sand with Clay Layer: $p - y$ Response
Single Pile in Clay with Sand Layer: $p - y$ Response

![Graph showing lateral pressure vs. lateral displacement for different depths.](image)

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$p - y$ Pressure Ratio Reduction for Layered Soils
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Pile Group Simulations
Bending Moments

![Graph showing bending moments for pile groups.](image-url)
Out of Plane Effects
Load Distribution per Pile

![Graph showing load distribution per pile with various rows and piles indicated.]
Piles Interaction at -2.0m ($\rho - y$)
Summary

- Importance of consistent formulation, material modeling and implementation
- Verified, validate models and simulations tools used for prediction of behavior
- Program and examples available in public domain (Author’s web site)