The Case for Probabilistic Elasto–Plasticity

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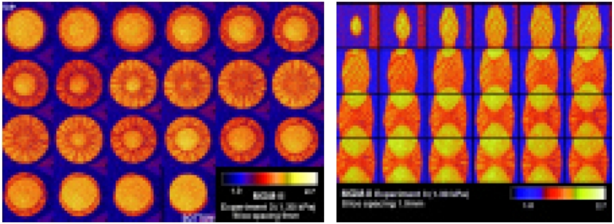
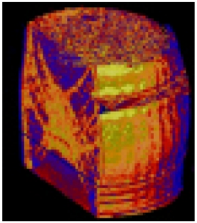
Outline

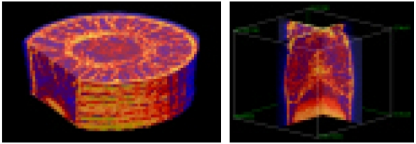
Outline

Failure Mechanisms for Geomaterials





Computed tomography (CT) images of resin-impregnated MGM specimens (above), are assembled to provide 3-D volume renderings (below) of density patterns formed by diffused bifurcation under the external loading stress profile applied during the experiments.



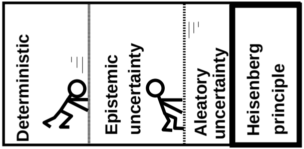
Soil: Inside Failure of "Uniform" MGM Specimen

Personal Motivation

- Probabilistic fish counting
- Williams' DEM simulations, differential displacement vortices
- SFEM round table
- Kavvas' probabilistic hydrology

Types of Uncertainties

- Epistemic uncertainty - due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools are not well developed
  - trade-off with aleatory uncertainty
- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools



Ergodicity

- Exchange ensemble averages for time averages
- Is soil elasto-plasticity ergodic?
  - Can soil elastic–plastic statistical properties be obtained by temporal averaging?
  - Will soil elastic–plastic statistical properties "renew" at each occurrence?
  - Are soil elastic–plastic statistical properties statistically independent?
- Claim in literature that structural nonlinear behavior is non–ergodic while earthquake characteristics are (?!)
- However, earthquake characteristics is representing mechanics (fault slip) on a different scale...

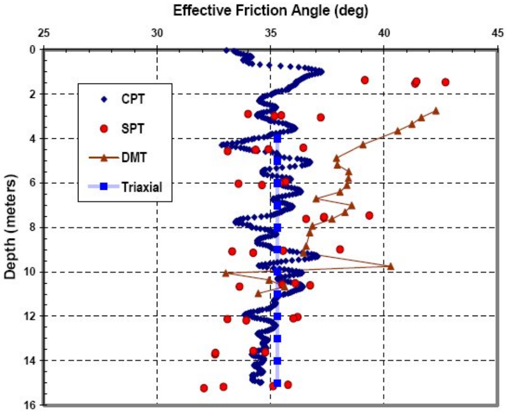
Historical Overview

- Brownian motion, Langevin equation $\rightarrow$ PDF governed by simple diffusion Eq. (Einstein 1905)
- With external forces $\rightarrow$ Fokker-Planck-Kolmogorov (FPK) for the PDF (Kolmogorov 1941)
- Approach for random forcing $\rightarrow$ relationship between the autocorrelation function and spectral density function (Wiener 1930)
- Approach for random coefficient $\rightarrow$ Functional integration approach (Hopf 1952), Averaged equation approach (Bharrucha-Reid 1968), Numerical approaches, Monte Carlo method

Outline

Material Behavior Inherently Uncertain

- Spatial variability
- Point-wise uncertainty, testing error, transformation error



(Mayne et al. (2000)

Motivation

Typical Coefficients of Variation of Different Soil Properties

Soil Property	Soil Type	PDF	Mean	COV (%)
Cone resistance	Sand Clay Clay	LN N/LN	*	*
Undrained shear strength	Clay (triaxial) Clay (index S_u) Clayey silt	LN LN N	*	5-20 10-35 5-15
Ratio S_u/σ'_{vo}	Clay	N/LN	*	5-15
Plastic limit	Clay	N	0.13-0.23	3-20
Liquid limit	Clay	N	0.30-0.80	3-20
Submerged unit weight	All soils	N	5-11 (kN/m^3)	0-10
Friction angle	Sand	N	*	2-5
Void ratio, porosity, initial void ratio	All soils	N	*	7-30
Over consolidation ratio	Clay	N/LN	*	10-35

(After Lacasse and Nadim 1996)

Soil Uncertainties and Quantification

- Natural variability of soil deposit (Fenton 1999)
  - Function of soil formation process
- Testing error (Stokoe et al. 2004)
  - Imperfection of instruments
  - Error in methods to register quantities
- Transformation error (Phoon and Kulhawy 1999)
  - Correlation by empirical data fitting (e.g. CPT data $\rightarrow$ friction angle etc.)

Probabilistic Material (Soil Site) Characterization

- Ideal: complete probabilistic site characterization
- Large (physically large but not statistically) amount of data
  - Site specific mean and coefficient of variation (COV)
  - Covariance structure from similar sites (e.g. Fenton 1999)
- Moderate amount of data $\rightarrow$ Bayesian updating (e.g. Phoon and Kulhawy 1999, Baecher and Christian 2003)
- Minimal data: general guidelines for typical sites and test methods (Phoon and Kulhawy (1999))
  - COVs and covariance structures of inherent variability
  - COVs of testing errors and transformation uncertainties.

Recent State-of-the-Art

- Governing equation
  - Dynamic problems $\rightarrow M\ddot{u} + C\dot{u} + Ku = \phi$
  - Static problems $\rightarrow Ku = \phi$
- Existing solution methods
  - **Random r.h.s** (external force random)
    - FPK equation approach
    - Use of fragility curves with deterministic FEM (DFEM)
  - **Random l.h.s** (material properties random)
    - Monte Carlo approach with DFEM $\rightarrow$ CPU expensive
    - Perturbation method $\rightarrow$ a linearized expansion! Error increases as a function of COV
    - Spectral method $\rightarrow$ developed for elastic materials so far
- New developments for elasto–plastic applications

Outline

Uncertainty Propagation through Constitutive Eq.

- Incremental el–pl constitutive equation $\dfrac{d\sigma_{ij}}{dt} = D_{ijkl}\dfrac{d\epsilon_{kl}}{dt}$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{for elastic} \\ \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} & \text{for elastic–plastic} \end{cases}$$

Previous Work

- Linear algebraic or differential equations $\rightarrow$ Analytical solution:
  - Variable Transf. Method (Montgomery and Runger 2003)
  - Cumulant Expansion Method (Gardiner 2004)
- Nonlinear differential equations (elasto-plastic/viscoelastic-viscoplastic):
  - Monte Carlo Simulation (Schueller 1997, De Lima et al 2001, Mellah et al. 2000, Griffiths et al. 2005...) $\rightarrow$ accurate, very costly
  - Perturbation Method (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 1997) $\rightarrow$ first and second order Taylor series expansion about mean - limited to problems with small C.O.V. and inherits "closure problem"

Problem Statement

- Incremental 3D elastic-plastic stress–strain:

$$\frac{d\sigma_{ij}}{dt} = \left\{ D^{el}_{ijkl} - \frac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} \right\} \frac{d\epsilon_{kl}}{dt}$$

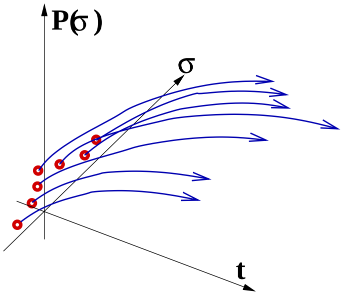
- Focus on 1D $\rightarrow$ a nonlinear ODE with random coefficient (material) and random forcing (ϵ)

$$\begin{aligned} \frac{d\sigma(x,t)}{dt} &= \beta(\sigma(x,t), D^{el}(x), q(x), r(x); x, t)\frac{d\epsilon(x,t)}{dt} \\ &= \eta(\sigma, D^{el}, q, r, \epsilon; x, t) \end{aligned}$$

with initial condition $\sigma(0) = \sigma_0$

Evolution of the Density $\rho(\sigma, t)$

- From each initial point in σ-space a trajectory starts out describing the corresponding solution of the stochastic process
- Movement of a cloud of initial points described by density $\rho(\sigma, 0)$ in σ-space, is governed by the constitutive equation,



Stochastic Continuity (Liouville) Equation

- phase density ρ of $\sigma(x,t)$ varies in time according to a continuity Liouville equation (Kubo 1963):

$$\frac{\partial \rho(\sigma(x,t),t)}{\partial t} = -\frac{\partial \eta(\sigma(x,t), D^{el}(x), q(x), r(x), \epsilon(x,t))}{\partial \sigma} \rho[\sigma(x,t),t]$$

- with initial conditions $\rho(\sigma, 0) = \delta(\sigma - \sigma_0)$

Ensemble Average form of Liouville Equation

Continuity equation written in ensemble average form (eg. cumulant expansion method (Kavvas and Karakas 1996)):

$$\frac{\partial \langle \rho(\sigma(x_t,t),t)\rangle}{\partial t} = -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))\right\rangle\right.\right.$$

$$+ \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))}{\partial \sigma};\right.$$

$$\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau},t-\tau)\right]\right\}\langle \rho(\sigma(x_t,t),t)\rangle\right]$$

$$+ \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t));\right.\right.\right.$$

$$\left.\left.\left.\eta(\sigma(x_{t-\tau},t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau},t-\tau))\right]\right\}\langle \rho(\sigma(x_t,t),t)\rangle\right]$$

Eulerian–Lagrangian FPK Equation

van Kampen's Lemma (van Kampen 1976) $\rightarrow < \rho(\sigma, t) >= P(\sigma, t)$, ensemble average of phase density is the probability density;

$$\frac{\partial P(\sigma(x_t, t), t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t, t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t, t)) \right\rangle\right.\right.$$

$$+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta(\sigma(x_t, t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t, t))}{\partial \sigma};\right.$$

$$\left.\left.\left.\eta(\sigma(x_{t-\tau}, t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)\right]\right\} P(\sigma(x_t, t), t)\right]$$

$$+ \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0 \left[\eta(\sigma(x_t, t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t, t));\right.\right.\right.$$

$$\left.\left.\left.\eta(\sigma(x_{t-\tau}, t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau))\right]\right\} P(\sigma(x_t, t), t)\right]$$

E–L FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)} P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)} P(\sigma,t)\right\}\right]$$

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- It is deterministic equation in probability density space
- It is linear PDE in probability density space → simplifies the numerical solution process

Template Solution of FPK Equation

- FPK diffusion–advection equation is applicable to any material model $\rightarrow$ only the coefficients $N_{(1)}$ and $N_{(2)}$ are different for different material models

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right] = -\frac{\partial \zeta}{\partial \sigma}$$

- Initial condition
  - Deterministic $\rightarrow$ Dirac delta function $\rightarrow P(\sigma,0) = \delta(\sigma)$
  - Random $\rightarrow$ Any given distribution
- Boundary condition: Reflecting BC $\rightarrow$ conserves probability mass $\zeta(\sigma,t)|_{At\ Boundaries} = 0$
- Finite Differences used for solution (among many others)

Application of FPK equation to Material Models

- FPK equation is applicable to any incremental elastic–plastic material model
- Solution in terms of PDF, not a single value of stress
- Influence of initial condition on the PDF of stress
- Mean stress yielding or
- Probabilistic yielding

Outline

Elastic Response with Random G

- General form of elastic constitutive rate equation

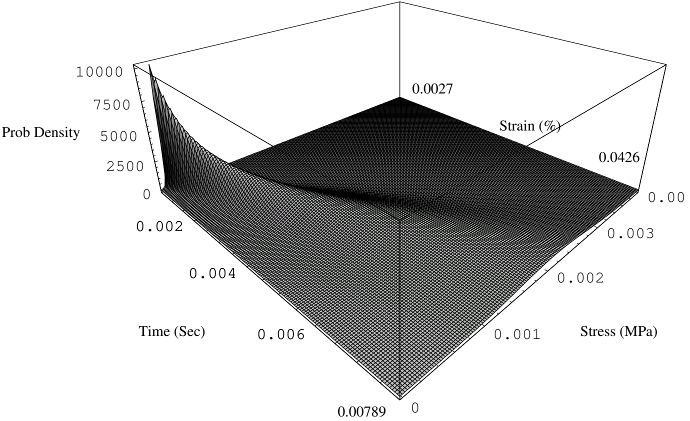
$$\frac{d\sigma_{12}}{dt} = 2G\frac{d\epsilon_{12}}{dt} = \eta(G, \epsilon_{12}; t)$$

- Advection and diffusion coefficients of FPK equation

$$N_{(1)} = 2\frac{d\epsilon_{12}}{dt} < G >$$

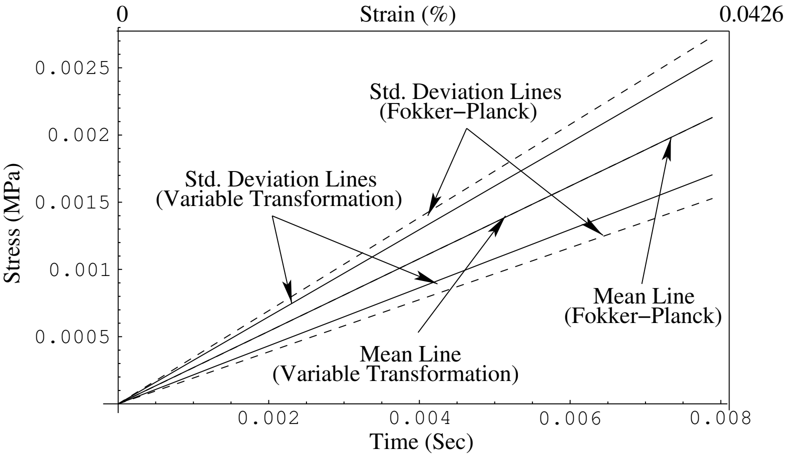
$$N_{(2)} = 4t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var[G]$$

Elastic Response with Random *G*



$< G >$ = 2.5 MPa; Std. Deviation[G] = 0.5 MPa

Verification – Variable Transformation Method



Modified Cam Clay Constitutive Model

$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, M, e_0, p_0, \lambda, \kappa, \epsilon_{12}; t)$$

$$\eta = \left[2G - \frac{\left(36\frac{G^2}{M^4}\right)\sigma_{12}^2}{\frac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\frac{G}{M^4}\right)\sigma_{12}^2 + \frac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)}\right]$$

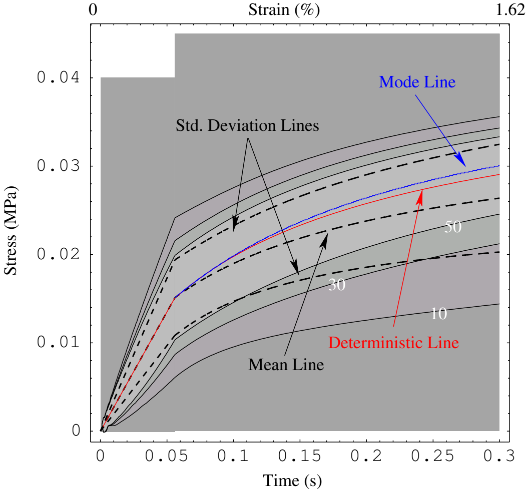
Advection and diffusion coefficients of FPK equation

$$N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov\left[\frac{\partial\eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau)\right]$$

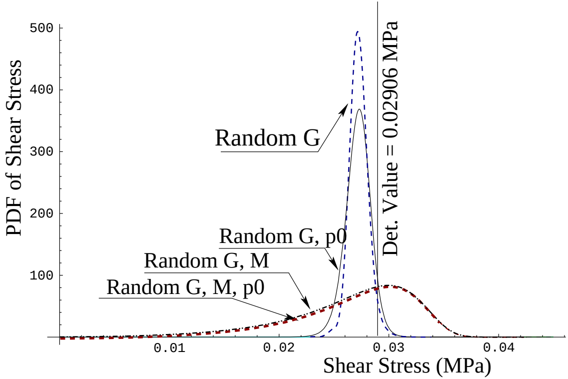
$$N_{(2)}^{(i)} = \int_0^t d\tau cov\left[\eta^{(i)}(t); \eta^{(i)}(t-\tau)\right]$$

Low OCR Cam Clay with Random G, M and p_0

- Non-symmetry in probability distribution
- Difference between mean, mode and deterministic
- Divergence at critical state because M is uncertain



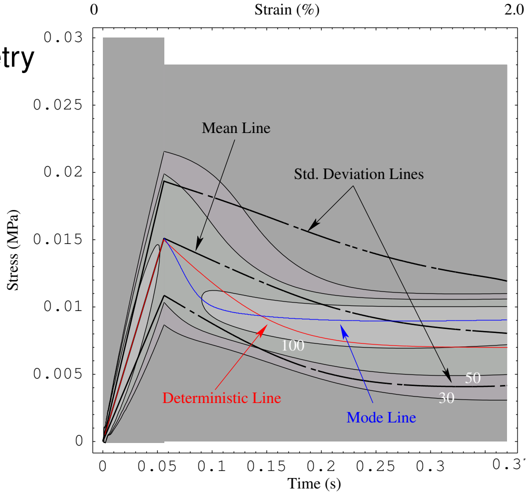
Comparison of Low OCR Cam Clay at ϵ = 1.62 %



- None coincides with deterministic
- Some very uncertain, some very certain
- Either on safe or unsafe side

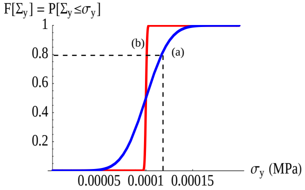
High OCR Cam Clay with Random *G* and *M*

- Large non-symmetry in probability distribution
- Significant differences in mean, mode, and deterministic
- Divergence at critical state, *M* is uncertain

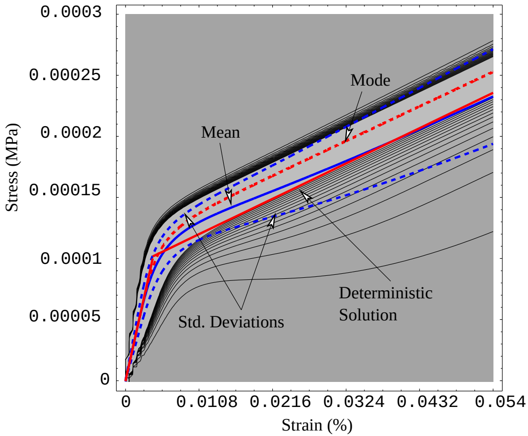


Probabilistic Yielding

- Weighted elastic and elastic–plastic Solution
$$\partial P(\sigma,t)/\partial t = -\partial\left(N^w_{(1)} P(\sigma,t) - \partial\left(N^w_{(2)} P(\sigma,t)\right)/\partial\sigma\right)/\partial\sigma$$
- Weighted advection and diffusion coefficients are then
$$N^w_{(1,2)}(\sigma) = (1 - P[\Sigma_y \le \sigma])N^{el}_{(1)} + P[\Sigma_y \le \sigma]N^{el-pl}_{(1)}$$
- Cumulative Probability Density function (CDF) of the yield function

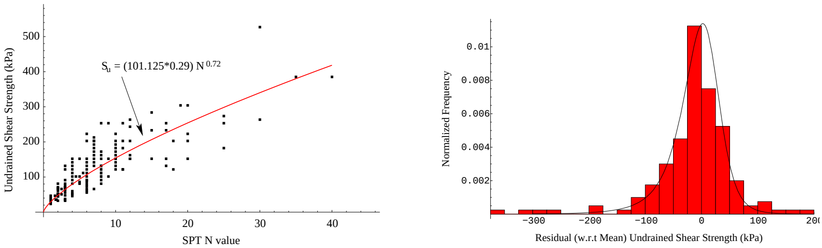


Transformation of a Bi–Linear (von Mises) Response



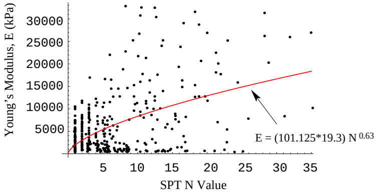
linear elastic – linear hardening plastic von Mises

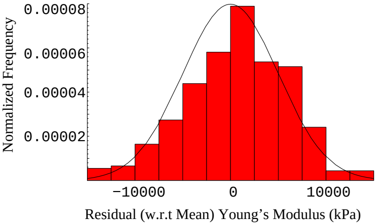
SPT Based Determination of Shear Strength



Transformation relationship between SPT *N*-value and undrained shear strength, s_u (cf. Phoon and Kulhawy (1999B)
Histogram of the residual (w.r.t the deterministic transformation equation) undrained strength, along with fitted probability density function

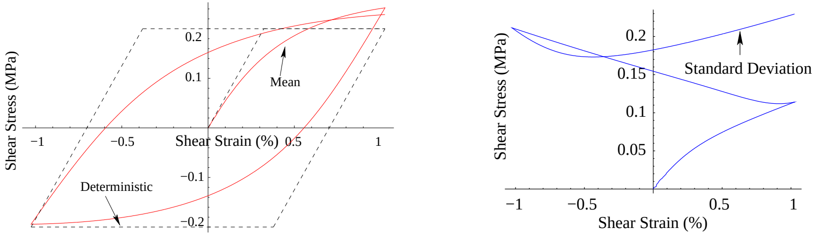
SPT Based Determination of Young's Modulus



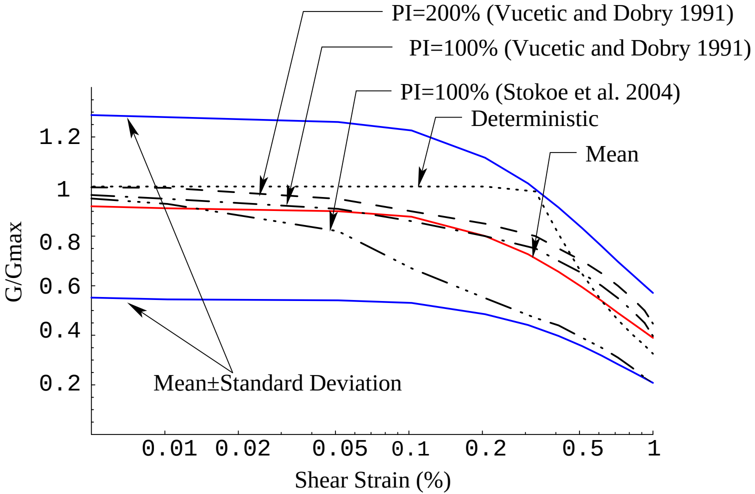


Transformation relationship between SPT *N*-value and pressure-meter Young's modulus, *E* (cf. Phoon and Kulhawy (1999B))
Histogram of the residual (w.r.t the deterministic transformation equation) Young's modulus, along with fitted probability density function

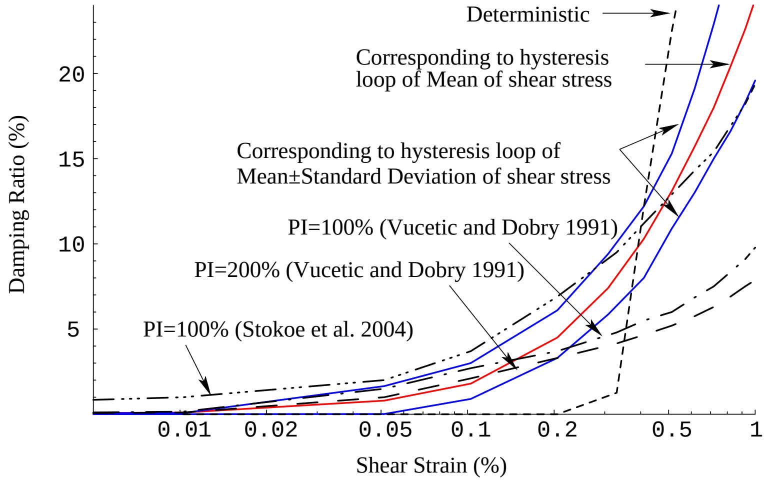
Cyclic Response of Such Uncertain Material



G/G_{max} Response



Damping Response



Outline

Governing Equations & Discretization Scheme

- Governing equations in geomechanics:

$$A\sigma = \phi(t); \quad Bu = \epsilon; \quad \sigma = D\epsilon$$

- Discretization (spatial and stochastic) schemes
  - Input random field material properties (D) $\rightarrow$ Karhunen–Loève (KL) expansion, optimal expansion, error minimizing property
  - Unknown solution random field (u) $\rightarrow$ Polynomial Chaos (PC) expansion
  - Deterministic spatial differential operators (A & B) $\rightarrow$ Regular shape function method with Galerkin scheme

Spectral Stochastic Elastic–Plastic FEM

- Minimizing norm of error of finite representation using Galerkin technique (Ghanem and Spanos 2003):

$$\sum_{n=1}^{N} K_{mn} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} C_{ijk} K'_{mnk} = \langle F_m \psi_i[\{\xi_r\}] \rangle$$

$$K_{mn} = \int_D B_n D B_m dV \qquad K'_{mnk} = \int_D B_n \sqrt{\lambda}_k h_k B_m dV$$

$$C_{ijk} = \langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \rangle \qquad F_m = \int_D \phi N_m dV$$

Inside SEPFEM

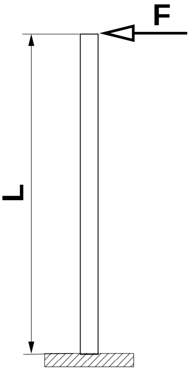
- Explicit stochastic elastic–plastic finite element computations
- FPK probabilistic constitutive integration at Gauss integration points
- Increase in (stochastic) dimensions (KL and PC) of the problem
- Development of the probabilistic elastic–plastic stiffness tensor

Outline

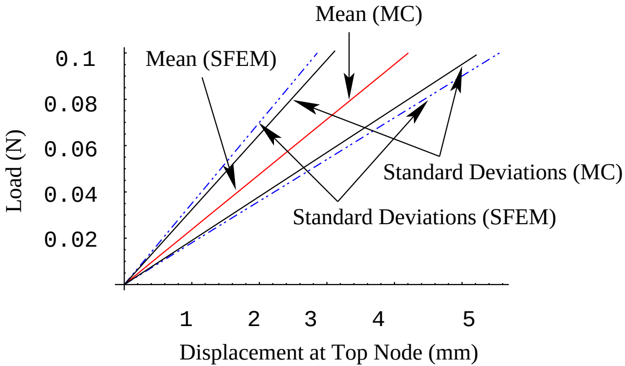
1–D Static Pushover Test Example

- Linear elastic model:
  $< G >= 2.5$ kPa,
  $Var[G] = 0.15\ \mathrm{kPa}^2$,
  correlation length for $G = 0.3$ m.

- Elastic–plastic material model,
  von Mises, linear hardening,
  $< G >= 2.5$ kPa,
  $Var[G] = 0.15\ \mathrm{kPa}^2$,
  correlation length for $G = 0.3$ m,
  $C_u = 5$ kPa,
  $C_u^{'} = 2$ kPa.

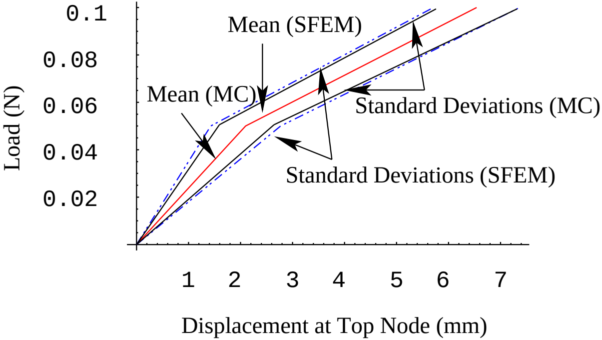


Linear Elastic FEM Verification



Mean and standard deviations of displacement at the top node, linear elastic material model,
KL-dimension=2, order of PC=2.

SEPFEM verification



Mean and standard deviations of displacement at the top node, von Mises elastic-plastic linear hardening material model, KL-dimension=2, order of PC=2.

Outline

Applications

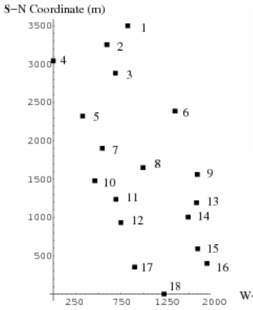
- Stochastic elastic–plastic simulations of soils and structures
- Probabilistic inverse problems
- Geotechnical site characterization design
- Optimal material design

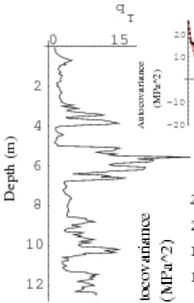
Seismic Wave Propagation through Stochastic Soil

- Soil as 12.5 m deep 1–D soil column (von Mises Material)
  - Properties (including testing uncertainty) obtained through random field modeling of CPT q_T
    $\langle q_T \rangle = 4.99\ MPa$; $Var[q_T] = 25.67\ MPa^2$;
    Cor. Length $[q_T] = 0.61\ m$; Testing Error $= 2.78\ MPa^2$
- q_T was transformed to obtain G: $G/(1-\nu) = 2.9 q_T$
  - Assumed transformation uncertainty = 5%
    $\langle G \rangle = 11.57 MPa$; $Var[G] = 142.32 MPa^2$
    Cor. Length $[G] = 0.61 m$
- Input motions: modified 1938 Imperial Valley

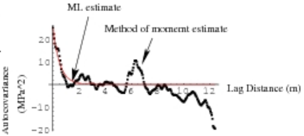
Random Field Parameters from Site Data

- Maximum likelihood estimates

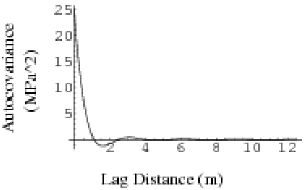




Typical CPT q_T

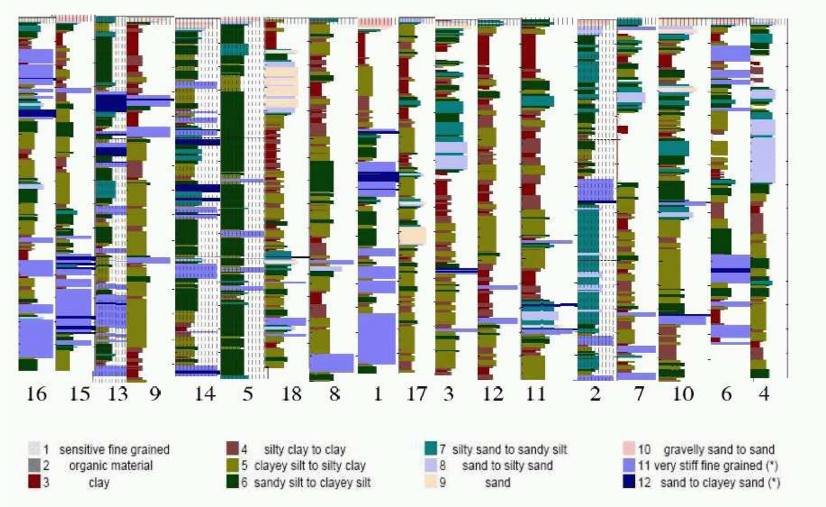


Finite Scale

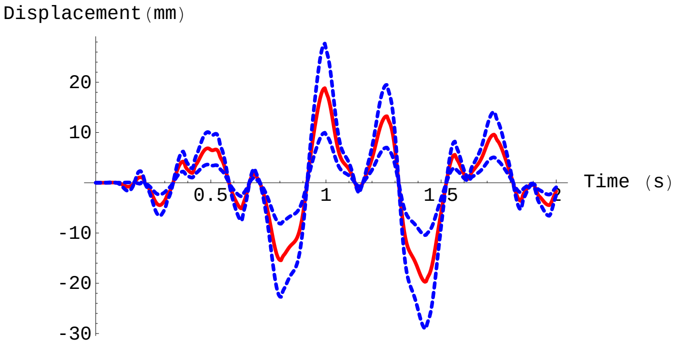


Fractal

"Uniform" CPT Site Data



Seismic Wave Propagation through Stochastic Soil



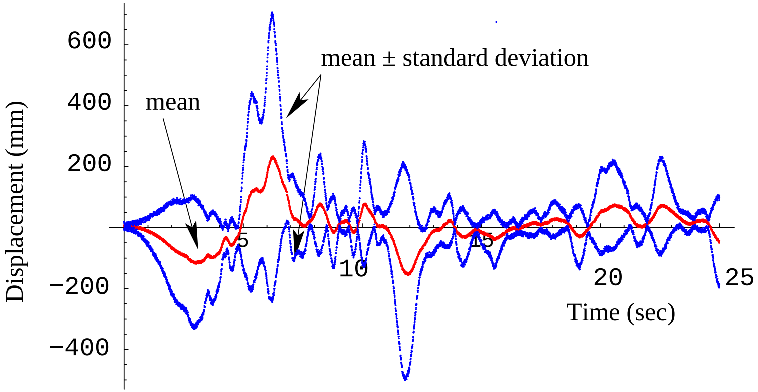
Mean± Standard Deviation

Outline

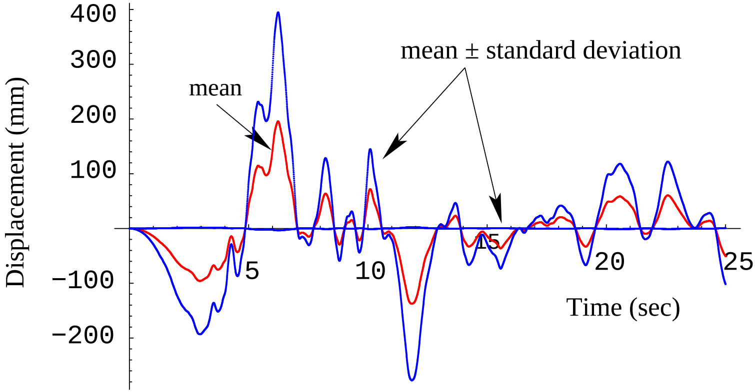
Three Approaches to Modeling

- Do nothing about site characterization (rely on experience): conservative **guess** of soil data, $COV = 225\%$, correlation length $= 12$m.
- Do better than standard site characterization: $COV = 103\%$, correlation length $= 0.61$m)
- Improve site characterization if probabilities of exceedance are unacceptable!

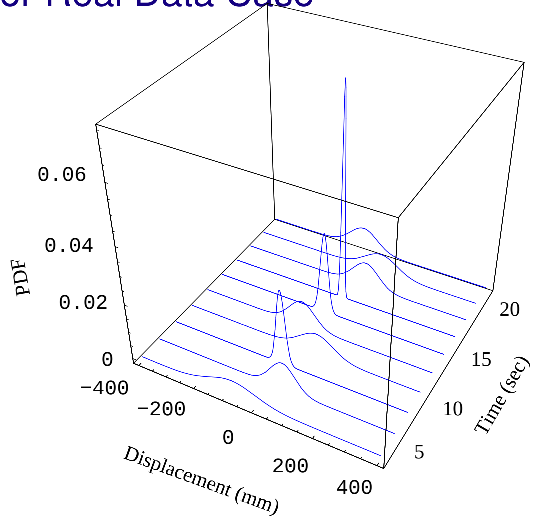
Evolution of Mean ± SD for Guess Case



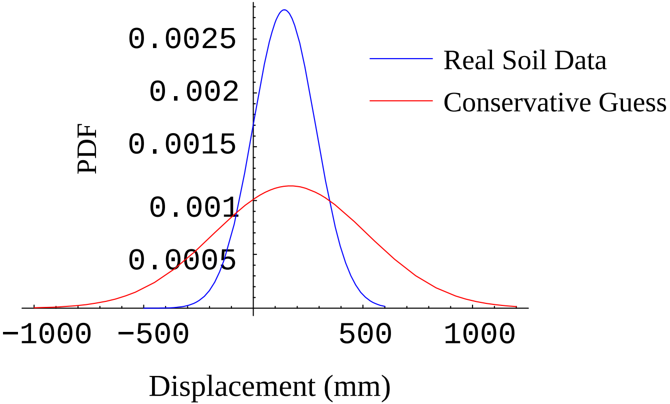
Evolution of Mean ± SD for Real Data Case



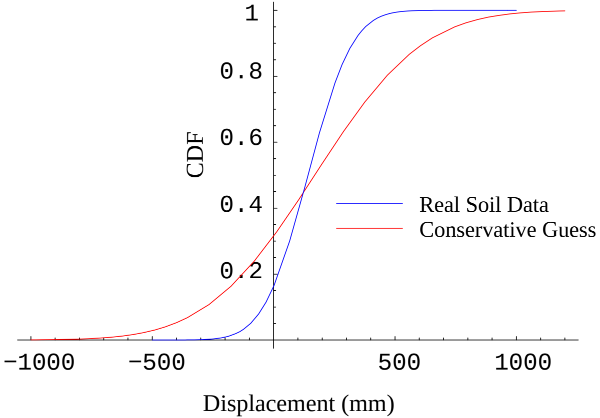
Full PDFs for Real Data Case



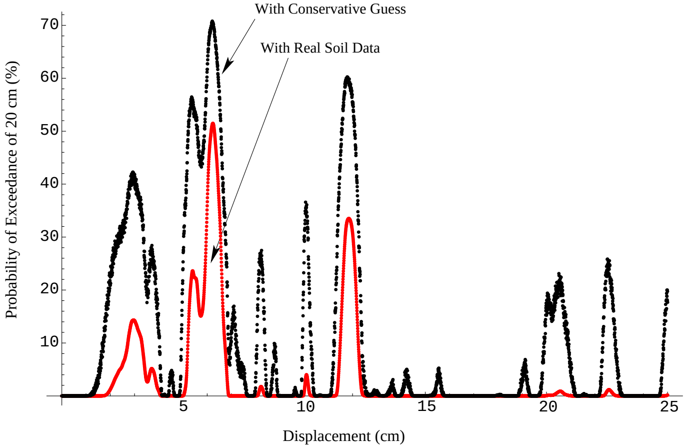
Example: PDF at 6 s



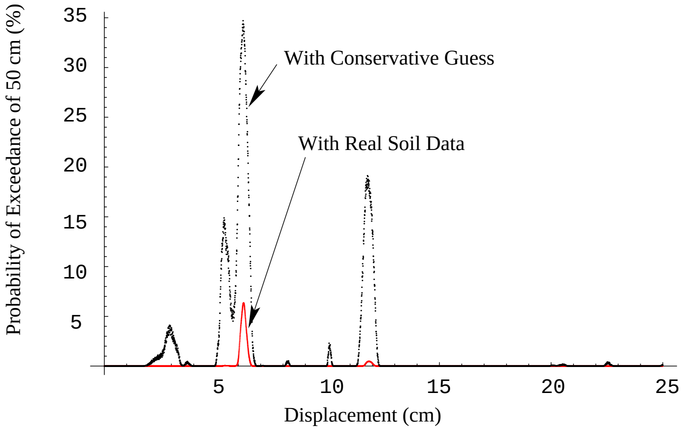
Example: CDF at 6 s



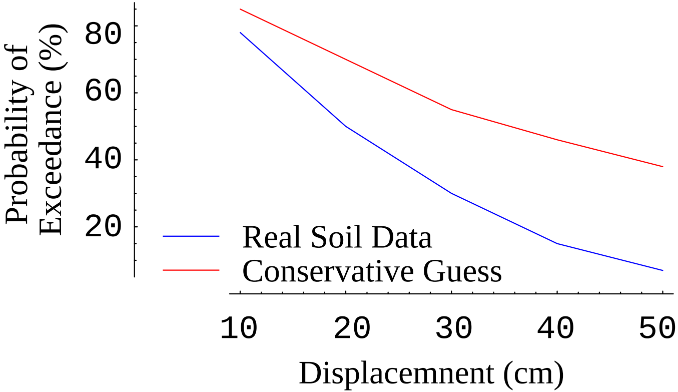
Probability of Exceedance of 20cm



Probability of Exceedance of 50cm



Probabilities of Exceedance vs. Displacements



Summary

- Behavior of materials is probably probabilistic!
- Technical developments are available and are being refined
- Human nature: how much do you want to know about potential problem?