Stochastic Elastic-Plastic Finite Element Method

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Outline

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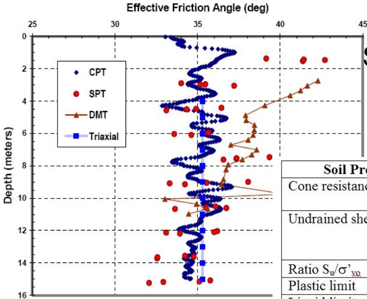
(Geo–) Materials are Inherently Uncertain



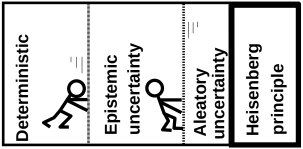
Spatial Variation of Friction Angle
(After Mayne et al. (2000))

Soil Property	Soil Type	PDF	Mean	COV (%)
Cone resistance	Sand Clay Clay	LN N/LN	*	*
Undrained shear strength	Clay (triaxial) Clay (index S_u) Clayey silt	LN LN N	*	5-20 10-35 5-15
Ratio S_u/σ'_{vo}	Clay	N/LN	*	5-15
Plastic limit	Clay	N	0.13-0.23	3-20
Liquid limit	Clay	N	0.30-0.80	3-20
Submerged unit weight	All soils	N	5-11 (kN/m^3)	0-10
Friction angle	Sand	N	*	2-5
Void ratio, porosity, initial void ratio	All soils	N	*	7-30
Over consolidation ratio	Clay	N/LN	*	10-35

Typical COVs of Different Soil Properties
(After Lacasse and Nadim 1996)

On Uncertainties

- Epistemic uncertainty - due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools not well developed, trade-off with aleatory uncertainty



- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools
- Ergodicity – exchange ensemble average for time average?
  - Applicable to soils, biomaterials, up for discussion for concrete, rock

Historical Overview

- Brownian motion, Langevin equation → PDF governed by simple diffusion Eq. (Einstein 1905)
- With external forces → Fokker-Planck-Kolmogorov (FPK) for the PDF (Kolmogorov 1941)
- Approach for random forcing → relationship between the autocorrelation function and spectral density function (Wiener 1930)
- Approach for random coefficient → Functional integration approach (Hopf 1952), Averaged equation approach (Bharrucha-Reid 1968), Monte Carlo method

Soil Uncertainties and Quantification

- Natural,spatial variability of soil deposit (Fenton 1999)
  - Function of soil formation process
- Testing (point–wise) error (Stokoe et al. 2004)
  - Imperfection of instruments
  - Error in methods to register quantities
- Transformation (point–wise) error (Phoon and Kulhawy 1999)
  - Correlation by empirical data fitting (e.g. CPT data → friction angle etc.)

Recent State-of-the-Art

- Governing equation
  - Dynamic problems → $M\ddot{u} + C\dot{u} + Ku = \phi$
  - Static problems → $Ku = \phi$
- Existing solution methods
  - **Random r.h.s** (external force random)
    - FPK equation approach
    - Use of fragility curves with deterministic FEM (DFEM)
  - **Random l.h.s** (material properties random)
    - Monte Carlo approach with DFEM → CPU expensive
    - Perturbation method → a linearized expansion! Error increases as a function of COV
    - Spectral method → developed for elastic materials so far
- Newly developed: Stochastic Elastic–Plastic Finite Element Method

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Constitutive Problem Setup

- 3D incremental elasto–plasticity:
$$d\sigma_{ij}/dt = \left\{ D^{el}_{ijkl} - \frac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} \right\} d\epsilon_{kl}/dt$$
- Phase density ρ of $\sigma(x, t)$ varies in time according to a continuity Liouville equation (Kubo 1963)
- Continuity equation written in ensemble average form (eg. cumulant expansion method (Kavvas and Karakas 1996))
- van Kampen's lemma (van Kampen 1976) $\rightarrow$ $< \rho(\sigma, t) >= P(\sigma, t)$, ensemble average of phase density is the probability density

Eulerian–Lagrangian FPK Equation

$$
\begin{aligned}
\frac{\partial P(\sigma(x_t,t),t)}{\partial t} &= -\frac{\partial}{\partial \sigma}\left[\left\{\left\langle \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))\right\rangle\right.\right.\\
&+ \int_0^t d\tau Cov_0\left[\frac{\partial \eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t))}{\partial \sigma};\right.\\
&\left.\left.\left.\eta(\sigma(x_{t-\tau}, t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau)\right]\right\} P(\sigma(x_t,t),t)\right]\\
&+ \frac{\partial^2}{\partial \sigma^2}\left[\left\{\int_0^t d\tau Cov_0\left[\eta(\sigma(x_t,t), D^{el}(x_t), q(x_t), r(x_t), \epsilon(x_t,t));\right.\right.\right.\\
&\left.\left.\left.\eta(\sigma(x_{t-\tau}, t-\tau), D^{el}(x_{t-\tau}), q(x_{t-\tau}), r(x_{t-\tau}), \epsilon(x_{t-\tau}, t-\tau))\right]\right\} P(\sigma(x_t,t),t)\right]
\end{aligned}
$$

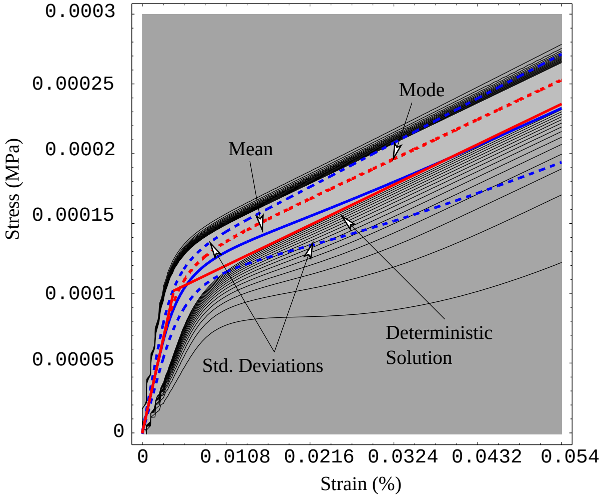
Euler–Lagrange FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma,t)}{\partial t} = -\frac{\partial}{\partial \sigma}\left[N_{(1)}P(\sigma,t) - \frac{\partial}{\partial \sigma}\left\{N_{(2)}P(\sigma,t)\right\}\right]$$

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- It is deterministic equation in probability density space
- It is linear PDE in probability density space → Simplifies the numerical solution process
- Template FPK diffusion–advection equation is applicable to any material model → only the coefficients $N_{(1)}$ and $N_{(2)}$ are different for different material models

Transformation of a Bi–Linear (von Mises) Response



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Stochastic Finite Element Formulation

- Governing equations:

$$A\sigma = \phi(t); \quad Bu = \epsilon; \quad \sigma = D\epsilon$$

- **Spatial** and **stochastic** discretization
  - Deterministic spatial differential operators (A & B) $\rightarrow$ Regular shape function method with Galerkin scheme
  - Input random field material properties (D) $\rightarrow$ Karhunen–Loève (KL) expansion, optimal expansion, error minimizing property
  - Unknown solution random field (u) $\rightarrow$ Polynomial Chaos (PC) expansion

Spectral Stochastic Elastic–Plastic FEM

- Minimizing norm of error of finite representation using Galerkin technique (Ghanem and Spanos 2003):

$$\sum_{n=1}^{N} K_{mn}^{ep} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} C_{ijk} K'^{ep}_{mnk} = \langle F_m \psi_i[\{\xi_r\}] \rangle$$

$$K_{mn}^{ep} = \int_D B_n D^{ep} B_m dV \qquad K'^{ep}_{mnk} = \int_D B_n \sqrt{\lambda}_k h_k B_m dV$$

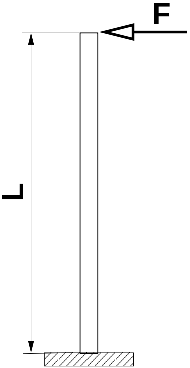
$$C_{ijk} = \langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \rangle \qquad F_m = \int_D \phi N_m dV$$

Inside SEPFEM

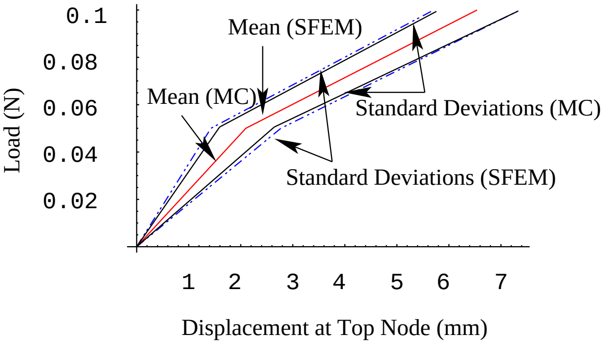
- Explicit stochastic elastic–plastic finite element computations
- FPK probabilistic constitutive integration at Gauss integration points
- Increase in (stochastic) dimensions (KL and PC) of the problem (parallelism)
- Development of the probabilistic elastic–plastic stiffness tensor

1–D Static Pushover Test Example

- Elastic–plastic material model, von Mises, linear hardening, $< G >= 2.5$ kPa, $Var[G] = 0.15\ \mathrm{kPa}^2$, correlation length for $G = 0.3$ m, $C_u = 5$ kPa, $C_u' = 2$ kPa.



SEPFEM verification



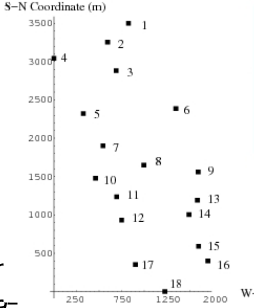
Mean and standard deviations of displacement at the top node, von Mises elastic-plastic linear hardening material model, KL-dimension=2, order of PC=2.

Outline

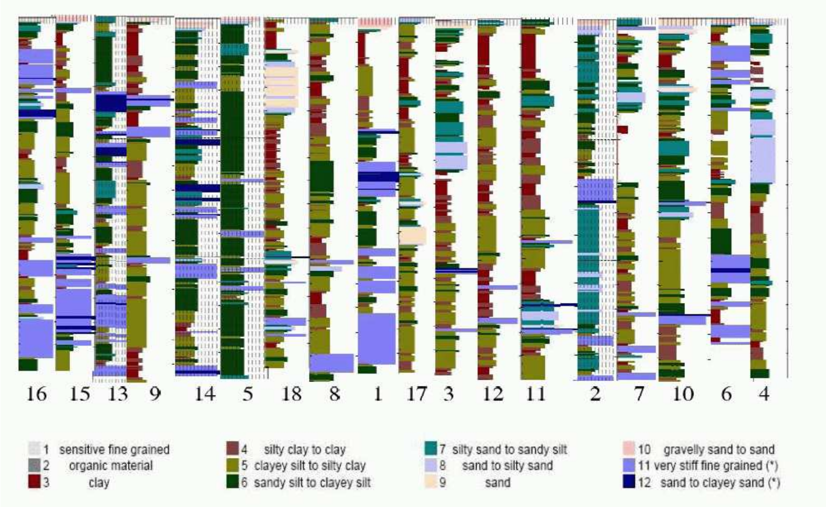
Seismic Wave Propagation Through Random Soil

Soil is 12.5 m deep 1–D soil column with random von Mises material:

- Shear modulus:
  $\langle G \rangle = 11.57 MPa$;
  $Var[G] = 142.32 MPa^2$;
  Cor. Length $[G] = 0.61$m
- Shear strength:
  $\langle q_T \rangle = 4.99\ MPa$;
  $Var[q_T] = 25.67\ MPa^2$;
  Cor. Length $[q_T] = 0.61$r
  Testing Error $= 2.78\ MPa$



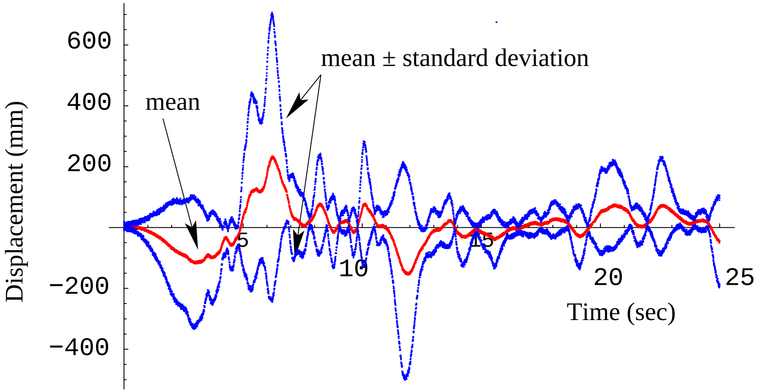
"Uniform" Soil Site



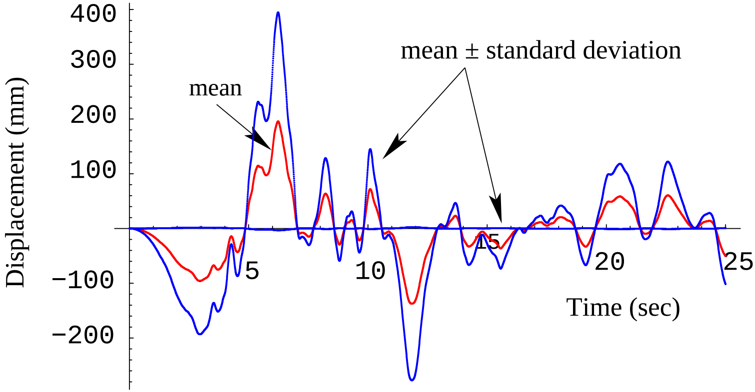
Three Approaches to Modeling

- Do nothing about site characterization (rely on experience): conservative **guess** of soil data, $COV = 225\%$, correlation length $= 12$m.
- Do better than standard site characterization: $COV = 103\%$, correlation length $= 0.61$m)
- Improve site (material) characterization if probabilities of exceedance are unacceptable!

Evolution of Mean ± SD for Guess Case

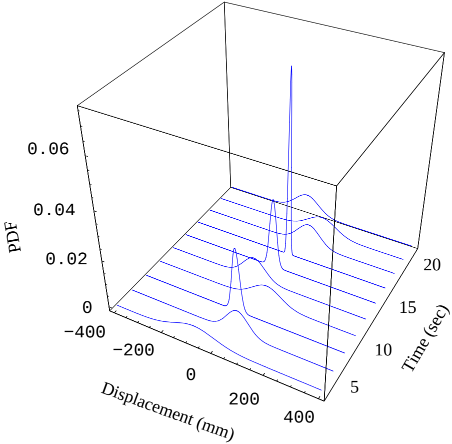


Evolution of Mean ± SD for Real Data Case

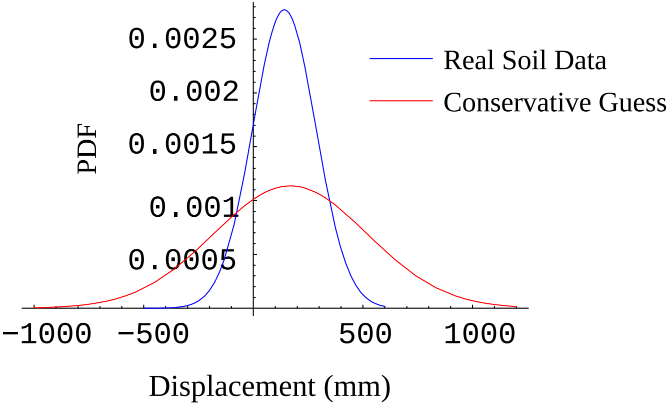


Full PDFs for Real Data Case

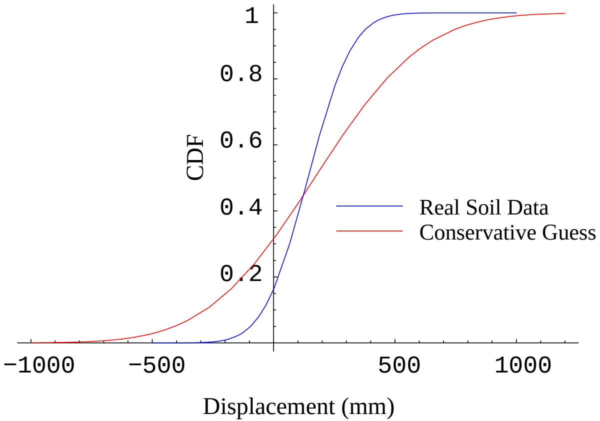
- PDF at the finite element nodes can be obtained using, e.g., Edgeworth expansion (Ghanem and Spanos 2003)
- Numerous applications, especially where extreme statistics are critical



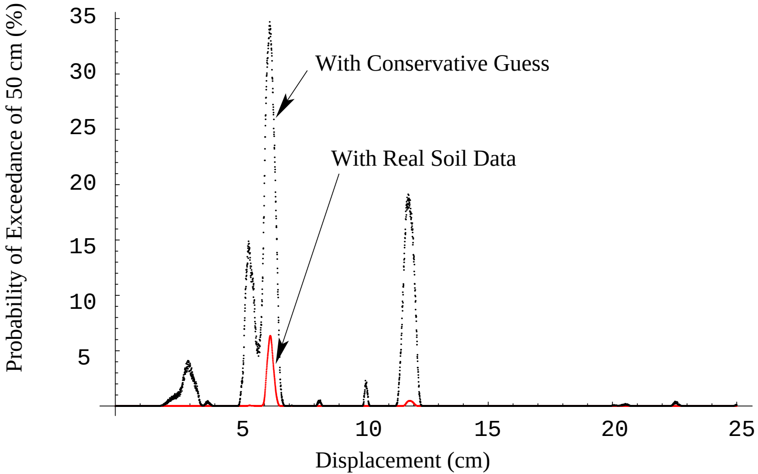
Example: PDF at 6 s



Example: CDF at 6 s



Probability of Exceedance of 50cm



Summary

- Behavior of materials is probably probabilistic, and one probably has to deal with it (in a probabilistic way)!
- Probabilistic Elasto–Plasticity and Stochastic Elastic–Plastic Finite Element Methodology has been developed and is being refined
- Human nature: how much do you want to know about potential problem?