Stochastic Elastic-Plastic Finite Element Method

Boris Jeremić

In collaboration with
Kallol Sett, You-Chen Chao and Levent Kavvas

CEE Dept., University of California, Davis
and
ESD, Lawrence Berkeley National Laboratory

Intel Corporation Webinar Series,
October 2010

Outline

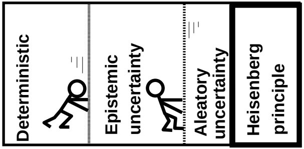
Outline

History

- Probabilistic fish counting
- Williams' DEM simulations, differential displacement vortexes
- SFEM round table
- Kavvas' probabilistic hydrology

Types of Uncertainties

- Epistemic uncertainty - due to lack of knowledge
  - Can be reduced by collecting more data
  - Mathematical tools are not well developed
  - trade-off with aleatory uncertainty
- Aleatory uncertainty - inherent variation of physical system
  - Can not be reduced
  - Has highly developed mathematical tools



Ergodicity

- Exchange ensemble averages for time averages
- Is elasto-plastic-damage response ergodic?
  - Can material elastic-plastic-damage statistical properties be obtained by temporal averaging?
  - Will elastic-plastic-damage statistical properties "renew" at each occurrence?
  - Are material elastic-plastic-damage statistical properties statistically independent?
- Important assumptions must be made and justified

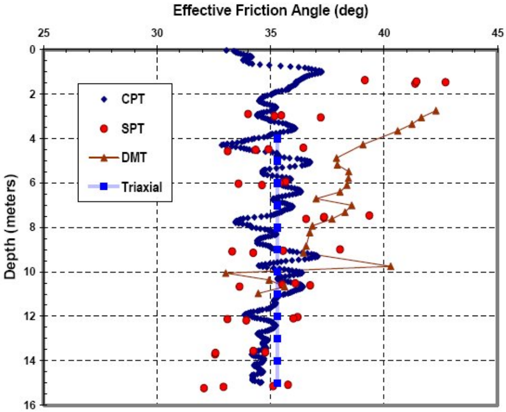
Historical Overview

- Brownian motion, Langevin equation → PDF governed by simple diffusion Eq. (Einstein 1905)
- With external forces → Fokker-Planck-Kolmogorov (FPK) for the PDF (Kolmogorov 1941)
- Approach for random forcing → relationship between the autocorrelation function and spectral density function (Wiener 1930)
- Approach for random coefficient → Functional integration approach (Hopf 1952), Averaged equation approach (Bharrucha-Reid 1968), Numerical approaches, Monte Carlo method

Outline

Material Behavior Inherently Uncertainties

- Spatial variability
- Point-wise uncertainty, testing error, transformation error



(Mayne et al. (2000)

Soil Uncertainties and Quantification

- Natural variability of material (for soil, Fenton 1999)
  - Function of material formation process
- Testing error (for soil, Stokoe et al. 2004)
  - Imperfection of instruments
  - Error in methods to register quantities
- Transformation error (for soil, Phoon and Kulhawy 1999)
  - Correlation by empirical data fitting (for soil, CPT data → friction angle etc.)

Probabilistic material (Soil Site) Characterization

- Ideal: complete probabilistic site characterization
- Large (physically large but not statistically) amount of data
  - Site specific mean and coefficient of variation (COV)
  - Covariance structure from similar sites (e.g. Fenton 1999)
- Moderate amount of data → Bayesian updating (e.g. Phoon and Kulhawy 1999, Baecher and Christian 2003)
- Minimal data: general guidelines for typical sites and test methods (Phoon and Kulhawy (1999))
  - COVs and covariance structures of inherent variability
  - COVs of testing errors and transformation uncertainties.

Recent State-of-the-Art

- Governing equation
  - Dynamic problems → $M\ddot{u} + C\dot{u} + Ku = \phi$
  - Static problems → $Ku = \phi$
- Existing solution methods
  - **Random r.h.s** (external force random)
    - FPK equation approach
    - Use of fragility curves with deterministic FEM (DFEM)
  - **Random l.h.s** (material properties random)
    - Monte Carlo approach with DFEM → CPU expensive
    - Perturbation method → a linearized expansion! Error increases as a function of COV
    - Spectral method → developed for elastic materials so far
- New developments for elasto-plastic-damage application

Outline

Uncertainty Propagation through Constitutive Eq.

- Incremental el–pl constitutive equation $\dfrac{d\sigma_{ij}}{dt} = D_{ijkl}\dfrac{d\epsilon_{kl}}{dt}$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{for elastic} \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} & \text{for elastic-plastic} \end{cases}$$

Previous Work

- Linear algebraic or differential equations → Analytical solution:
  - Variable Transf. Method (Montgomery and Runger 2003)
  - Cumulant Expansion Method (Gardiner 2004)
- Nonlinear differential equations (elasto-plastic/viscoelastic-viscoplastic):
  - Monte Carlo Simulation (Schueller 1997, De Lima et al 2001, Mellah et al. 2000, Griffiths et al. 2005...) → accurate, very costly
  - Perturbation Method (Anders and Hori 2000, Kleiber and Hien 1992, Matthies et al. 1997) → first and second order Taylor series expansion about mean - limited to problems with small C.O.V. and inherits "closure problem"

Problem Statement

- Incremental 3D elastic-plastic-damage stress–strain:

$$\frac{d\sigma_{ij}}{dt} = \left\{ D^{el}_{ijkl} - \frac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} \right\} \frac{d\epsilon_{kl}}{dt}$$

- Focus on 1D $\rightarrow$ a nonlinear ODE with random coefficient (material) and random forcing (ϵ)

$$\begin{aligned} \frac{d\sigma(x,t)}{dt} &= \beta(\sigma(x,t), D^{el}(x), q(x), r(x); x, t) \frac{d\epsilon(x,t)}{dt} \\ &= \eta(\sigma, D^{el}, q, r, \epsilon; x, t) \end{aligned}$$

with initial condition $\sigma(0) = \sigma_0$

Solution to Probabilistic Elastic-Plastic Problem

- Use of stochastic continuity (Liouiville) equation (Kubo 1963)
- With cumulant expansion method (Kavvas and Karakas 1996)
- To obtain ensemble average form of Liouville Equation
- Which, with van Kampen's Lemma (van Kampen 1976): ensemble average of phase density is the probability density
- Yields Eulerian-Lagrangian form of the Fokker-Planck-Kolmogorov equation

Eulerian–Lagrangian FPK Equation

$$\frac{\partial P(\sigma_{ij}(x_t,t),t)}{\partial t} = \frac{\partial}{\partial \sigma_{mn}} \left[\left\{ \left\langle \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)) \right\rangle \right.\right.$$

$$+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t))}{\partial \sigma_{ab}}; \right.$$

$$\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau) \right] \right\} P(\sigma_{ij}(x_t,t),t) \Big]$$

$$+ \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[\left\{ \int_0^t d\tau Cov_0 \left[\eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)); \right.\right.\right.$$

$$\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right]$$

Eulerian–Lagrangian FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma_{ij}, t)}{\partial t} = -\frac{\partial}{\partial \sigma_{mn}} \left[N_{(1)} P(\sigma_{ij}, t) - \frac{\partial}{\partial \sigma_{ab}} \left\{ N_{(2)} P(\sigma_{ij}, t) \right\} \right]$$

- Complete probabilistic description of response
- Solution PDF is second-order exact to covariance of time (exact mean and variance)
- Deterministic equation in probability density space
- Linear PDE in probability density space → simplifies the numerical solution process
- Applicable to any elastic-plastic-damage material model (only coefficients $N_{(1)}$ and $N_{(2)}$ differ)

3D FPK Equation

$$\frac{\partial P(\sigma_{ij}(x_t,t),t)}{\partial t} = \frac{\partial}{\partial \sigma_{mn}} \left[\left\{ \left\langle \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)) \right\rangle \right. \right.$$
$$+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t))}{\partial \sigma_{ab}}; \right.$$
$$\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau) \right] \right\} P(\sigma_{ij}(x_t,t),t) \Big]$$
$$+ \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[\left\{ \int_0^t d\tau Cov_0 \left[\eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)); \right. \right. \right.$$
$$\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right]$$

- 6 equations, 3D probabilistic stress-strain response
- 3 equations, 3D in principal stress-strain space
- 2 equations, 2D in principal stress-strain space

Outline

Elastic Response with Random G

- General form of elastic constitutive rate equation

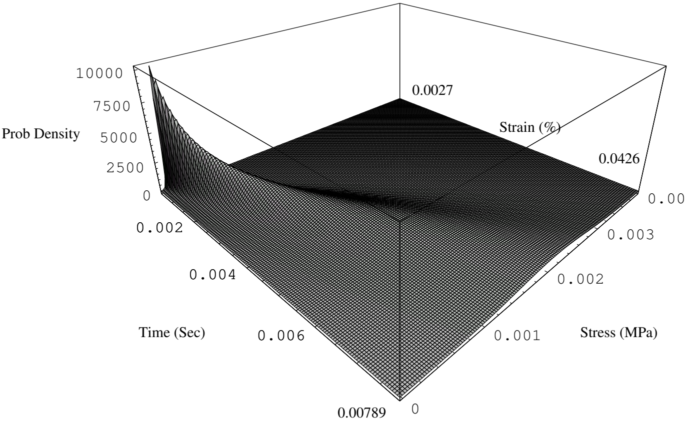
$$\frac{d\sigma_{12}}{dt} = 2G\frac{d\epsilon_{12}}{dt} = \eta(G, \epsilon_{12}; t)$$

- Advection and diffusion coefficients of FPK equation

$$N_{(1)} = 2\frac{d\epsilon_{12}}{dt} < G >$$

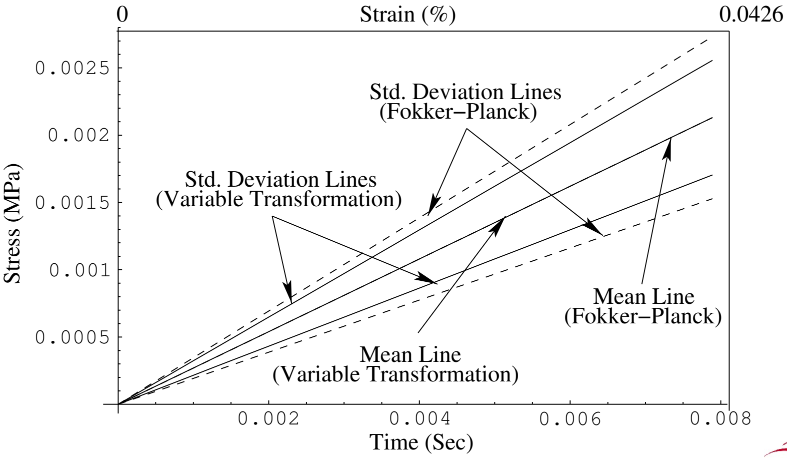
$$N_{(2)} = 4t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var[G]$$

Elastic Response with Random *G*



$< G >$ = 2.5 MPa; Std. Deviation$[G]$ = 0.5 MPa

Verification – Variable Transformation Method



Drucker-Prager Linear Hardening with Random G

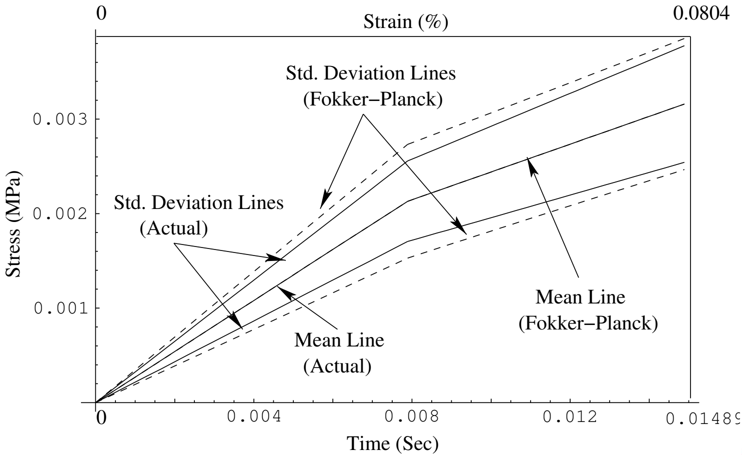
$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, K, \alpha, \alpha', \epsilon_{12}; t)$$

Advection and diffusion coefficients of FPK equation

$$N_{(1)} = \frac{d\epsilon_{12}}{dt}\left\langle 2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right\rangle$$

$$N_{(2)} = t\left(\frac{d\epsilon_{12}}{dt}\right)^2 Var\left[2G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1\alpha'}\right]$$

Verification of D–P E–P Response - Monte Carlo



Modified Cam Clay Constitutive Model

$$\frac{d\sigma_{12}}{dt} = G^{ep}\frac{d\epsilon_{12}}{dt} = \eta(\sigma_{12}, G, M, e_0, p_0, \lambda, \kappa, \epsilon_{12}; t)$$

$$\eta = \left[2G - \frac{\left(36\dfrac{G^2}{M^4}\right)\sigma_{12}^2}{\dfrac{(1+e_0)p(2p-p_0)^2}{\kappa} + \left(18\dfrac{G}{M^4}\right)\sigma_{12}^2 + \dfrac{1+e_0}{\lambda-\kappa}pp_0(2p-p_0)}\right]$$

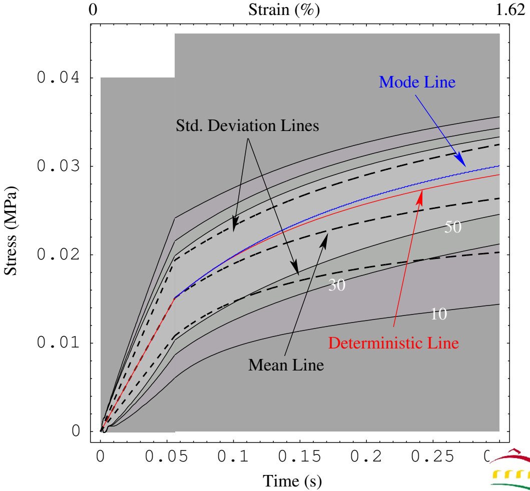
Advection and diffusion coefficients of FPK equation

$$N_{(1)}^{(i)} = \left\langle \eta^{(i)}(t) \right\rangle + \int_0^t d\tau cov\left[\frac{\partial\eta^{(i)}(t)}{\partial t}; \eta^{(i)}(t-\tau)\right]$$

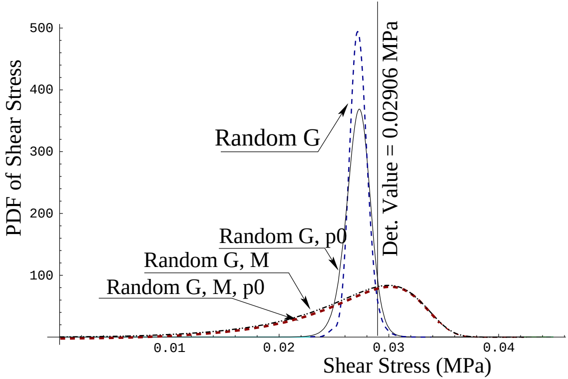
$$N_{(2)}^{(i)} = \int_0^t d\tau cov\left[\eta^{(i)}(t); \eta^{(i)}(t-\tau)\right]$$

Low OCR Cam Clay with Random G, M and p_0

- Non-symmetry in probability distribution
- Difference between mean, mode and deterministic
- Divergence at critical state because M is uncertain



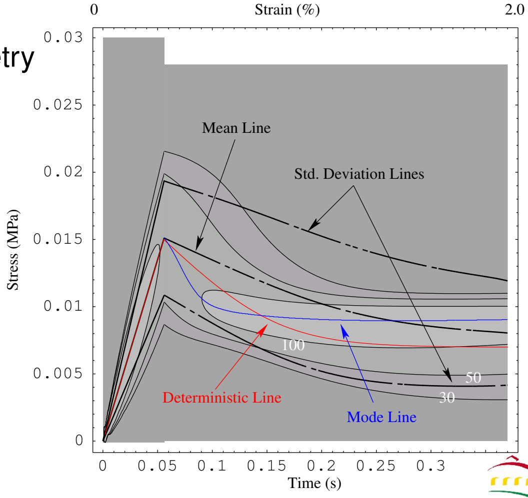
Comparison of Low OCR Cam Clay at ϵ = 1.62 %



- None coincides with deterministic
- Some very uncertain, some very certain
- Either on safe or unsafe side

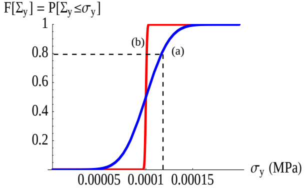
High OCR Cam Clay with Random *G* and *M*

- Large non-symmetry in probability distribution
- Significant differences in mean, mode, and deterministic
- Divergence at critical state, *M* is uncertain

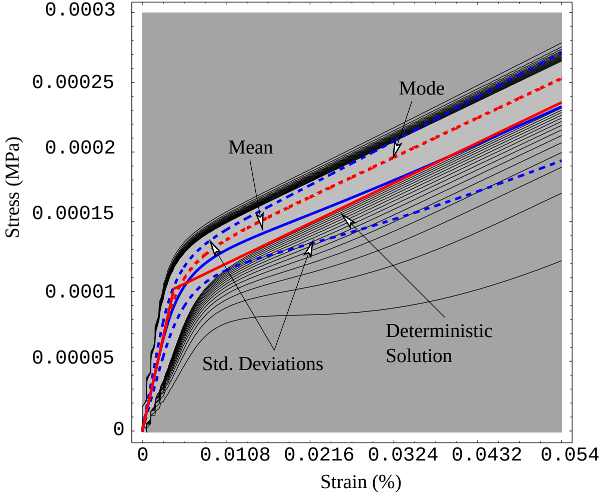


Probabilistic Yielding

- Weighted elastic and elastic-plastic Solution
  $\partial P(\sigma,t)/\partial t = -\partial \left(N^{w}_{(1)} P(\sigma,t) - \partial \left(N^{w}_{(2)} P(\sigma,t) \right)/\partial\sigma \right)/\partial\sigma$
- Weighted advection and diffusion coefficients are then
  $N^{w}_{(1,2)}(\sigma) = (1 - P[\Sigma_y \leq \sigma])N^{el}_{(1)} + P[\Sigma_y \leq \sigma]N^{el-pl}_{(1)}$
- Cumulative Probability Density function (CDF) of the yield function



Transformation of a Bi–Linear (von Mises) Response



Outline

Governing Equations & Discretization Scheme

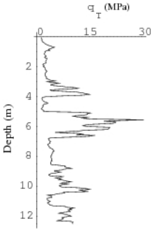
- Governing equations of mechanics:

$$A\sigma = \phi(t); \quad Bu = \epsilon; \quad \sigma = D\epsilon$$

- Discretization (spatial and stochastic) schemes
  - Input random field material properties (D) $\rightarrow$ Karhunen–Loève (KL) expansion, optimal expansion, error minimizing property
  - Unknown solution random field (u) $\rightarrow$ Polynomial Chaos (PC) expansion
  - Deterministic spatial differential operators (A & B) $\rightarrow$ Regular shape function method with Galerkin scheme

Truncated Karhunen–Loève (KL) expansion

- Representation of input random fields in eigen-modes of covariance kernel



$$q_T(x,\theta) = \bar{q}_T(x) + \sum_{n=1}^{M} \sqrt{\lambda_n}\xi_n(\theta) f_n(x)$$

$$\int_D C(x_1, x_2) f(x_2) dx_2 = \lambda f(x_1)$$

$$\xi_i(\theta) = \frac{1}{\sqrt{\lambda_i}} \int_D \left[q_T(x,\theta) - \bar{q}_T(x) \right] f_i(x) dx$$

- Error minimizing property
- Optimal expansion $\rightarrow$ minimization of number of stochastic dimensions

Polynomial Chaos (PC) Expansion

- Covariance kernel is not known a priori

$$u(x,\theta) = \sum_{j=1}^{L} e_j \chi_j(\theta) b_j(x)$$

- Can be expressed as functional of known random variables and unknown deterministic function

$$u(x,\theta) = \zeta[\xi_i(\theta), x]$$

- Need a basis of known random variables $\rightarrow$ PC expansion

$$\chi_j(\theta) = \sum_{i=0}^{P} \gamma_i^{(j)} \psi_i \left[\{\xi_r\}\right]$$

$$u(x,\theta) = \sum_{j=1}^{L} \sum_{i=0}^{P} \gamma_i^{(j)} \psi_i[\{\xi_r\}] e_j b_j(x) = \sum_{i=0}^{P} \psi_i[\{\xi_r\}] d_i(x)$$

Spectral Stochastic Elastic–Plastic FEM

- Minimizing norm of error of finite representation using Galerkin technique (Ghanem and Spanos 2003):

$$\sum_{n=1}^{N} K_{mn} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} C_{ijk} K'_{mnk} = \langle F_m \psi_i[\{\xi_r\}] \rangle$$

$$K_{mn} = \int_D B_n D B_m dV \qquad K'_{mnk} = \int_D B_n \sqrt{\lambda}_k h_k B_m dV$$

$$C_{ijk} = \left\langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \right\rangle \qquad F_m = \int_D \phi N_m dV$$

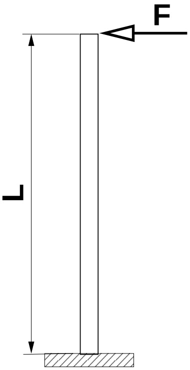
Inside SEPFEM

- Explicit stochastic elastic-plastic-damage finite element computations
- FPK probabilistic constitutive integration at Gauss integration points
- Increase in (stochastic) dimensions (KL and PC) of the problem (parallelism)
- Development of the probabilistic elastic-plastic-damage stiffness tensor

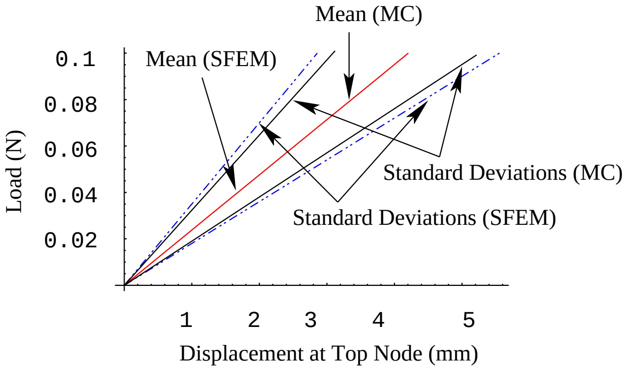
Outline

1–D Static Pushover Test Example

- Linear elastic model:
  $< G >= 2.5$ kPa,
  $Var[G] = 0.15 \text{ kPa}^2$,
  correlation length for $G = 0.3$ m.
- Elastic–plastic material model, von Mises, linear hardening,
  $< G >= 2.5$ kPa,
  $Var[G] = 0.15 \text{ kPa}^2$,
  correlation length for $G = 0.3$ m,
  $C_u = 5$ kPa,
  $C_u' = 2$ kPa.

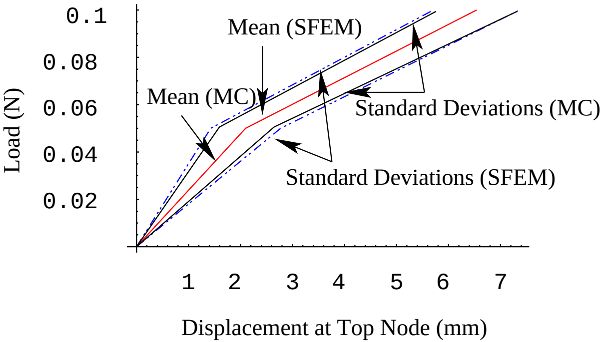


Linear Elastic FEM Verification



Mean and standard deviations of displacement at the top node, linear elastic material model, KL-dimension=2, order of PC=2.

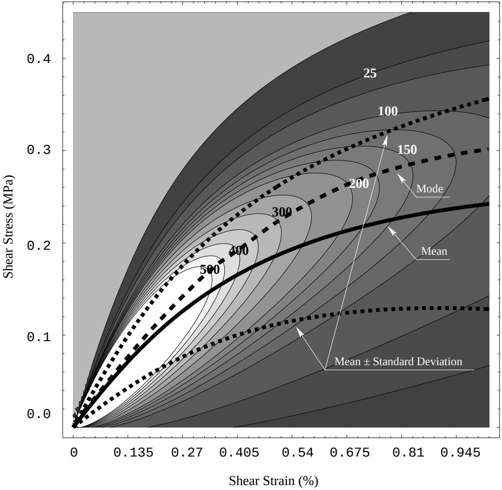
SEPFEM verification



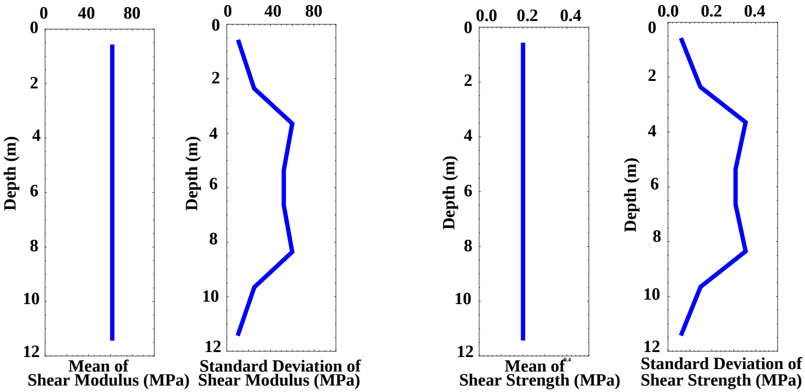
Mean and standard deviations of displacement at the top node, von Mises elastic-plastic linear hardening material model, KL-dimension=2, order of PC=2.

Outline

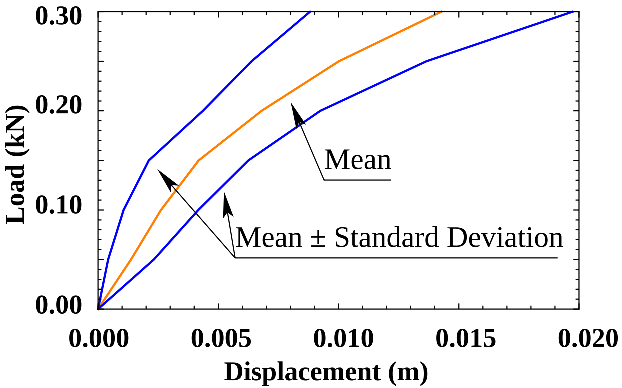
Probabilistic Material Response



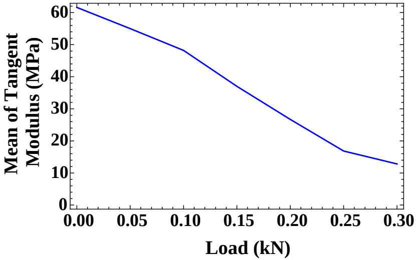
Elasticity and Strength, Mean and Std.Dev.

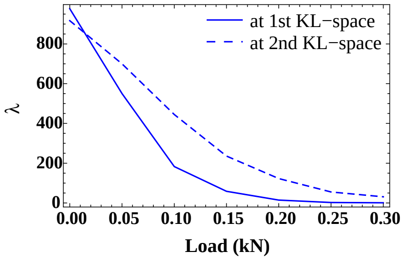


Displacement Results at the Top

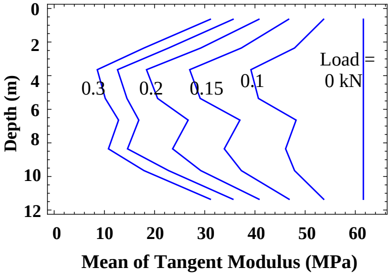


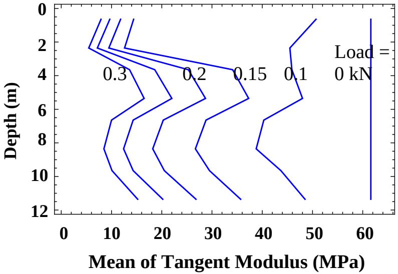
Statistics of the Tangent Stiffness: Mean and λ for the 1st and 2nd KL-spaces



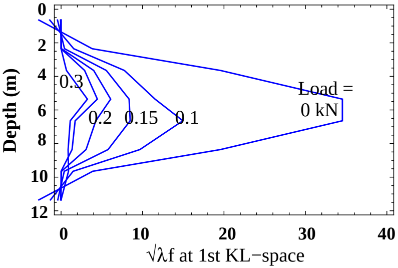


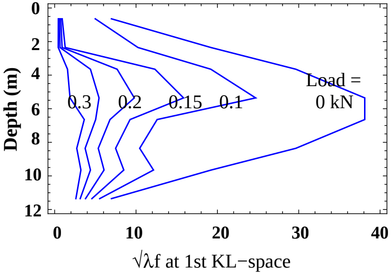
Mean Stiffness Evolution, Cor.Len. 0.1m, 1.0m



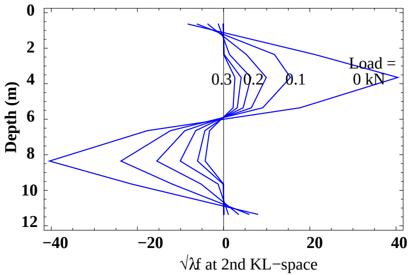


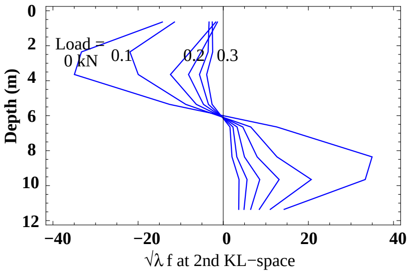
Mean Prob. Stiffness ($\sqrt{\lambda_1}\ f_1$) Evolution, Cor.Len. 0.1m, 1.0m



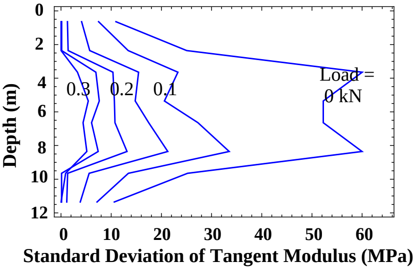


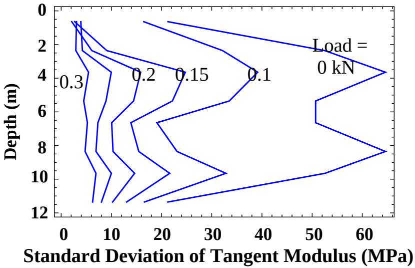
Mean Prob. Stiffness ($\sqrt{\lambda_2}\ f_2$) Evolution, Cor.Len. 0.1m, 1.0m



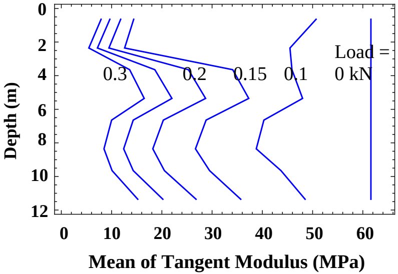


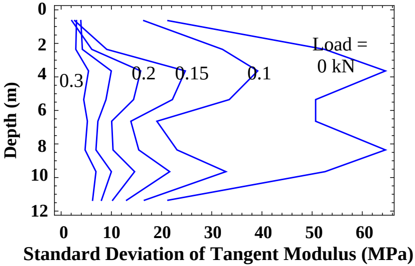
Std.Dev Stiffness Evolution, Cor.Len. 0.1m, 1.0m





Probability for Softening (?), Cor.Len. 1.0m





Outline

Uncertain Dynamics

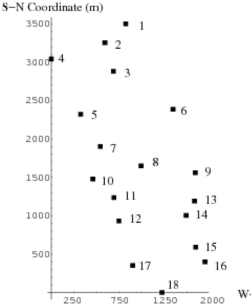
- Stochastic elastic–plastic simulations of soils and structures
- Probabilistic inverse problems
- Geotechnical site characterization design
- Optimal material design

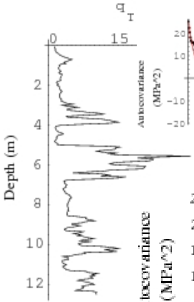
Decision About Site (Material) Characterization

- Do an inadequate site characterization (rely on experience): conservative **guess** for soil data, $COV = 225\%$, large correlation length (length of a model).
- Do a good site characterization: $COV = 103\%$, correlation length calculated ($= 0.61$m)
- Do an excellent (much improved) site characterization if probabilities of exceedance are unacceptable!

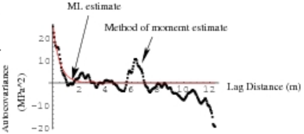
Random Field Parameters from Site Data

- Maximum likelihood estimates

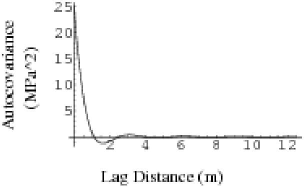




Typical CPT q_T

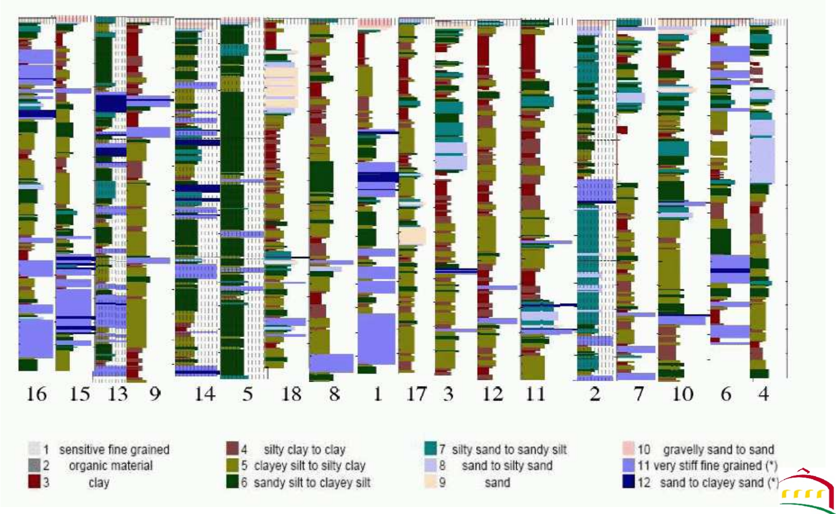


Finite Scale



Fractal

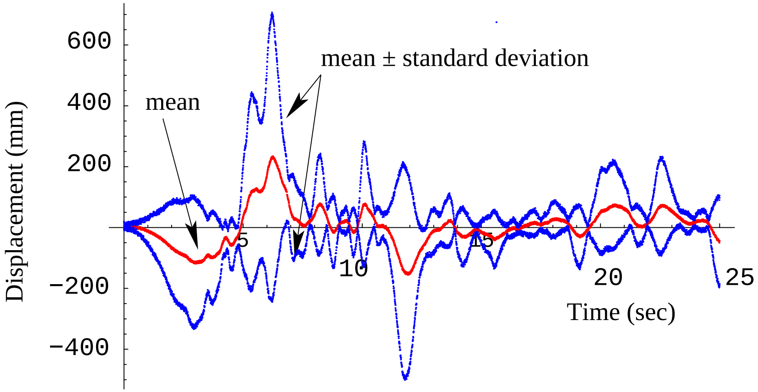
"Uniform" CPT Site Data (Courtesy of USGS)



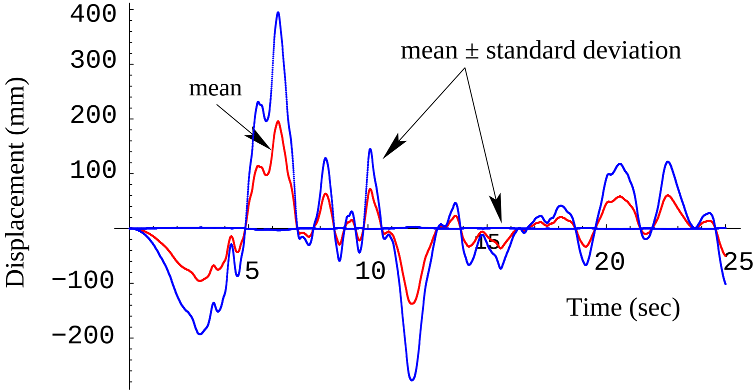
Statistics of Stochastic Soil Profile(s)

- Soil as 12.5m deep 1–D soil column (von Mises material)
  - Properties (including testing uncertainty) obtained through random field modeling of CPT q_T
    $\langle q_T \rangle = 4.99\ MPa$; $Var[q_T] = 25.67\ MPa^2$;
    Cor. Length $[q_T] = 0.61\ m$; Testing Error $= 2.78\ MPa^2$
- q_T was transformed to obtain G: $G/(1-\nu) = 2.9 q_T$
  - Assumed transformation uncertainty = 5%
    $\langle G \rangle = 11.57 MPa$; $Var[G] = 142.32 MPa^2$
    Cor. Length $[G] = 0.61 m$
- Input motions: modified 1938 Imperial Valley

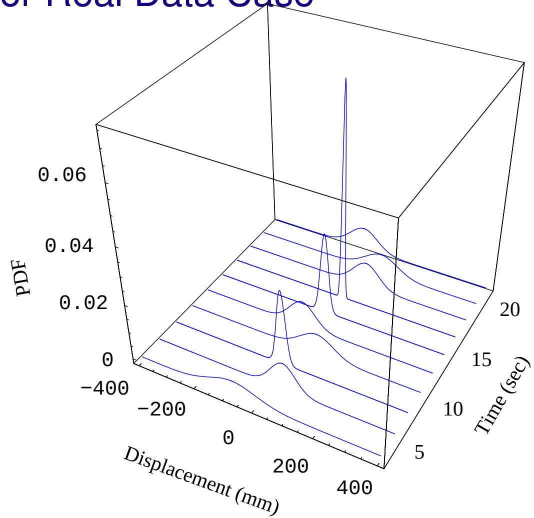
Evolution of Mean ± SD for Guess Case



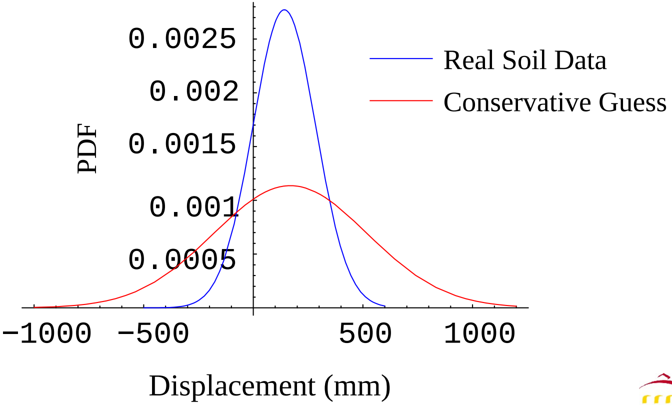
Evolution of Mean ± SD for Real Data Case



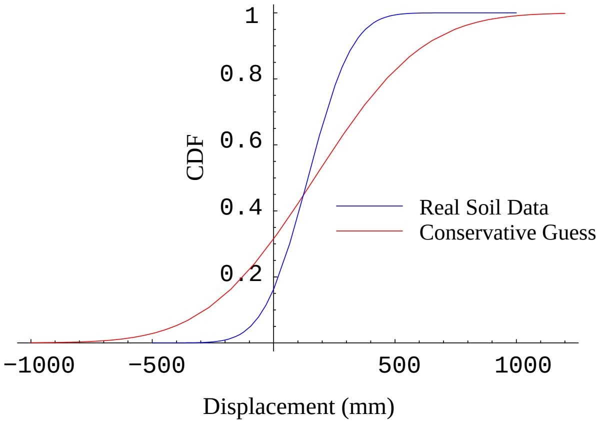
Full PDFs for Real Data Case



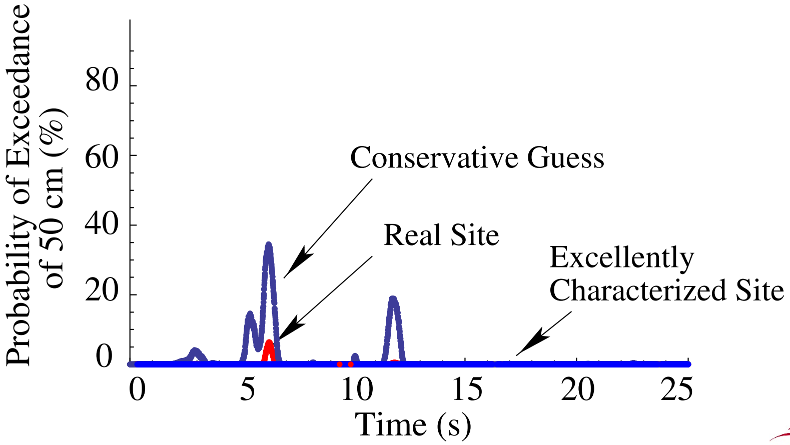
Example: PDF at 6 s



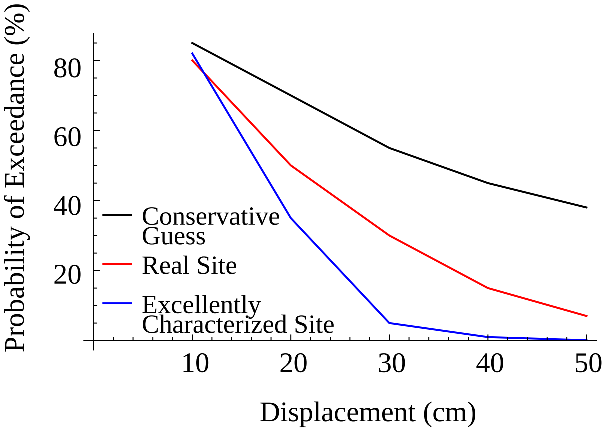
Example: CDF at 6 s



Probability of Unacceptable Deformation (50cm)



Risk Informed Decision Process



Summary

- Second-order (mean and variance) exact, method for simulations of solids with probabilistic elastic-plastic-damage material and stochastic loading
- Input:
  - elastic-plastic-damage point-wise uncertainty (PDFs of material paramaters, LHS),
  - spatial uncertainty (KL eigen modes, LHS) and
  - uncertain loading (RHS)
- Output: accurate PDFs and CDFs of displacements, velocities, accelerations, stress, strain...
- Probably numerous applications

Thank You