Stochastic Elastic-Plastic Finite Element Method for Performance Risk Simulations

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Outline

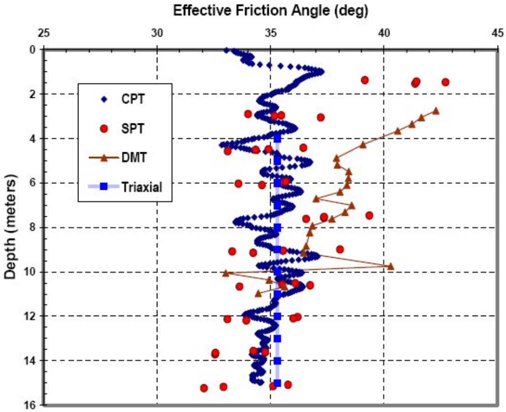
Determining Risk for Civil Engineering Object Behavior

- Risk: inherent, intrinsic, constitutive part of civil engineering
- Uncertain loads (!)
- Uncertain materials (!!)
- Uncertain human factor (!)

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$$

Material Behavior Inherently Uncertain

- Spatial variability
- Point-wise uncertainty
  - testing error
  - transformation error



(Mayne et al. (2000)

Outline

Uncertainty Propagation through Constitutive Eq.

- Incremental el–pl constitutive equation $\frac{d\sigma_{ij}}{dt} = D_{ijkl}\frac{d\epsilon_{kl}}{dt}$

$$D_{ijkl} = \begin{cases} D^{el}_{ijkl} & \text{for elastic} \\ D^{el}_{ijkl} - \dfrac{D^{el}_{ijmn} m_{mn} n_{pq} D^{el}_{pqkl}}{n_{rs} D^{el}_{rstu} m_{tu} - \xi_* r_*} & \text{for elastic–plastic} \end{cases}$$

- What if all (any) material parameters are uncertain
- Since material **is** inherently spatially variable and uncertain at the point, PEP and SEPFEM methods were developed

Solution to Probabilistic Elastic-Plastic Problem

- Use of stochastic continuity (Liouiville) equation (Kubo 1963)
- With cumulant expansion method (Kavvas and Karakas 1996)
- To obtain ensemble average form of Liouville Equation
- Which, with van Kampen's Lemma (van Kampen 1976): ensemble average of phase density is the probability density
- Yields Eulerian-Lagrangian form of the Forward Kolmogorov (Fokker-Planck-Kolmogorov) equation

Eulerian–Lagrangian FPK Equation

$$
\begin{aligned}
\frac{\partial P(\sigma_{ij}(x_t,t),t)}{\partial t} &= \frac{\partial}{\partial \sigma_{mn}} \left[\left\{ \left\langle \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)) \right\rangle \right.\right. \\
&+ \int_0^t d\tau Cov_0 \left[\frac{\partial \eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t))}{\partial \sigma_{ab}}; \right. \\
&\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)\right] \right\} P(\sigma_{ij}(x_t,t),t) \right] \\
&+ \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[\left\{ \int_0^t d\tau Cov_0 \left[\eta_{mn}(\sigma_{mn}(x_t,t), D_{mnrs}(x_t), \epsilon_{rs}(x_t,t)); \right.\right.\right. \\
&\left.\left.\left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t,t),t) \right]
\end{aligned}
$$

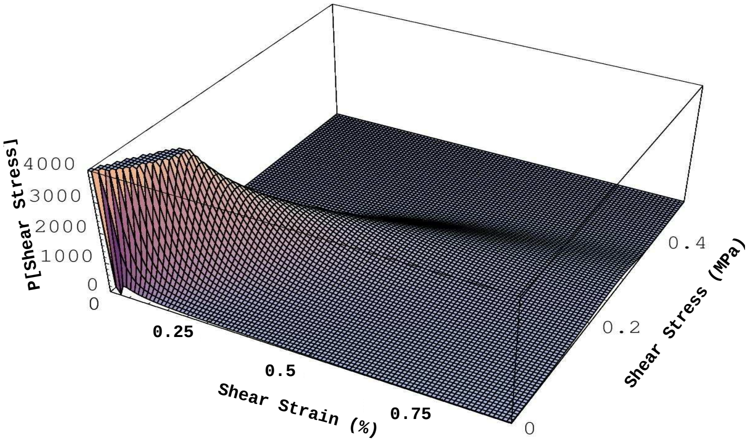
Eulerian–Lagrangian FPK Equation

- Advection-diffusion equation

$$\frac{\partial P(\sigma_{ij},t)}{\partial t} = -\frac{\partial}{\partial \sigma_{ab}} \left[N_{ab}^{(1)} P(\sigma_{ij},t) - \frac{\partial}{\partial \sigma_{cd}} \left\{ N_{abcd}^{(2)} P(\sigma_{ij},t) \right\} \right]$$

- Complete probabilistic description of response
- Solution PDF is **second-order exact** to covariance of time (exact mean and variance)
- Deterministic equation in probability density space
- Linear PDE in probability density space $\rightarrow$ simplifies the numerical solution process
- Applicable to any elastic-plastic-damage material model (only coefficients $N_{ab}^{(1)}$ and $N_{abcd}^{(2)}$ differ)

Probabilistic Elastic-Plastic Response



Outline

Governing Equations & Discretization Scheme

- Governing equations:

$$A\sigma = \phi(t) \qquad Bu = \epsilon \qquad \sigma = \mathbf{D}\epsilon$$

- Discretization (spatial and stochastic) schemes
  - Input random field material properties (**D**) $\rightarrow$ Karhunen–Loève (KL) expansion, optimal expansion, error minimizing property
  - Unknown solution random field (u) $\rightarrow$ Polynomial Chaos (PC) expansion
  - Deterministic spatial differential operators (A & B) $\rightarrow$ Regular shape function method with Galerkin scheme

Spectral Stochastic Elastic–Plastic FEM

- Minimizing norm of error of finite representation using Galerkin technique (Ghanem and Spanos 2003):

$$\sum_{n=1}^{N} K_{mn} d_{ni} + \sum_{n=1}^{N} \sum_{j=0}^{P} d_{nj} \sum_{k=1}^{M} C_{ijk} K'_{mnk} = \langle F_m \psi_i[\{\xi_r\}] \rangle$$

$$K_{mn} = \int_D B_n \mathbf{D} B_m dV \qquad K'_{mnk} = \int_D B_n \sqrt{\lambda_k} h_k B_m dV$$

$$C_{ijk} = \langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \rangle \qquad F_m = \int_D \phi N_m dV$$

Inside SEPFEM

- Stochastic elastic–plastic (explicit) finite element computations
- FPK probabilistic constitutive integration at Gauss integration points
- Increase in (stochastic) dimensions (KL and PC) of the problem
- Development of the probabilistic elastic–plastic stiffness tensor

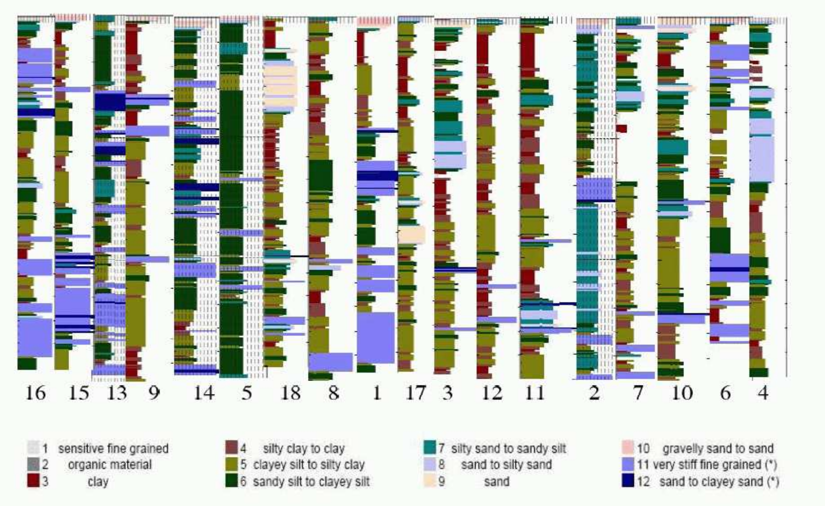
Outline

Risk Assessment Applications

- Any problem ($\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$) with known
    - PDFs of material parameters,
    - PDFs of loading

  can be analyzed using PEP and SEPFEM to obtain PDFs of DOFs, stress, strain...
- PEP solution is second order accurate (exact mean and standard deviation)
- SEPFEM solution (PDFs) can be made as accurate as need be
- Tails of PDFs can than be used to develop accurate risk
- Application to a realistic case of seismic wave propagation

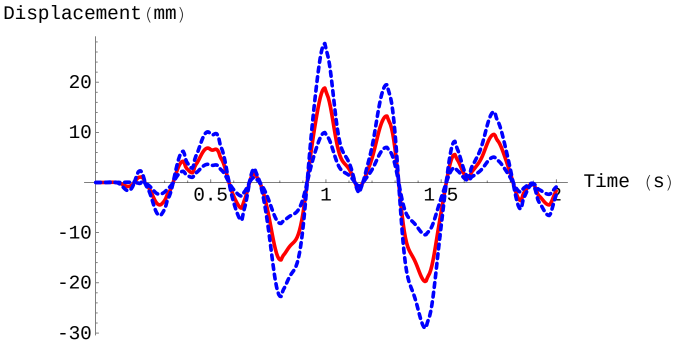
"Uniform" CPT Site Data



Seismic Wave Propagation through Stochastic Soil

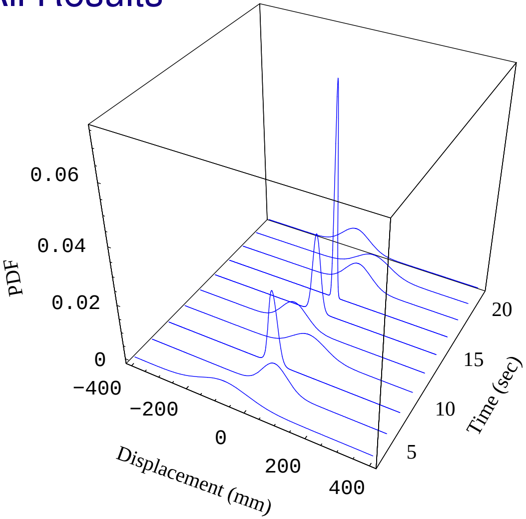
- Soil as 12.5 m deep 1–D soil column (von Mises Material)
  - Properties (including testing uncertainty) obtained through random field modeling of CPT q_T
    $\langle q_T \rangle = 4.99\ MPa$; $Var[q_T] = 25.67\ MPa^2$;
    Cor. Length $[q_T] = 0.61\ m$; Testing Error $= 2.78\ MPa^2$
- q_T was transformed to obtain G: $G/(1-\nu) = 2.9 q_T$
  - Assumed transformation uncertainty = 5%
    $\langle G \rangle = 11.57 MPa$; $Var[G] = 142.32 MPa^2$
    Cor. Length $[G] = 0.61 m$
- Input motions: modified 1938 Imperial Valley

Seismic Wave Propagation through Stochastic Soil



Mean± Standard Deviation

Full PDFs All Results

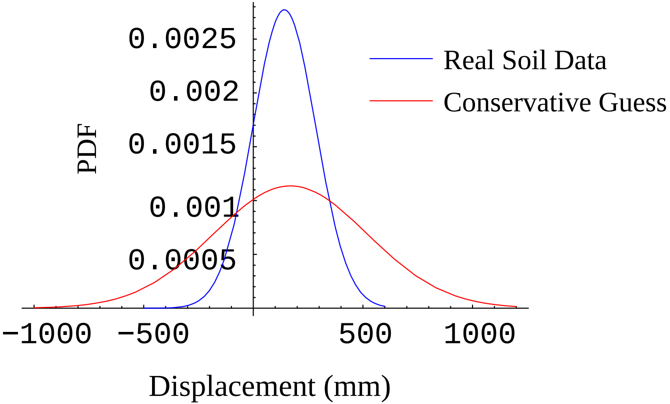


Outline

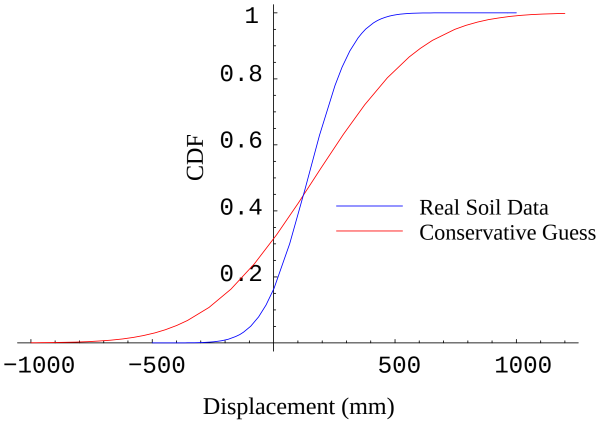
Example: Three Approaches to Modeling

- **Do nothing** about site (material) characterization (rely on experience): conservative guess for soil data, $COV = 225\%$, correlation length $= 12$m.
- **Do better** than standard site (material) characterization: $COV = 103\%$, correlation length $= 0.61$m)
- **Do the best** site (material) characterization to reduce probabilities of exceedance

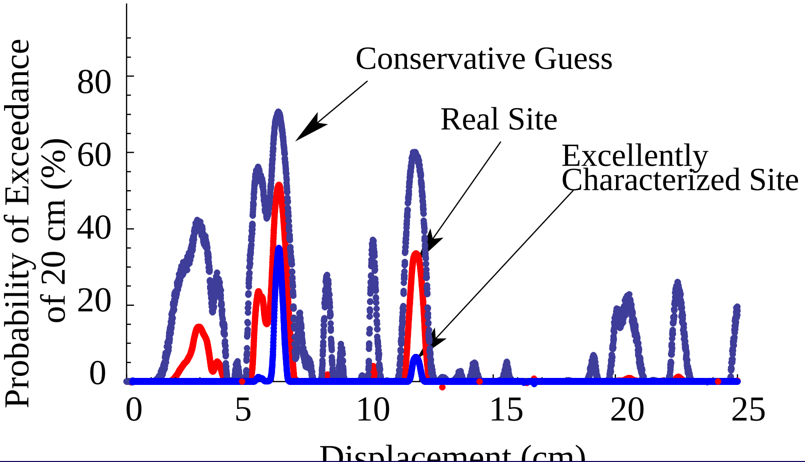
Example: PDF at 6 s



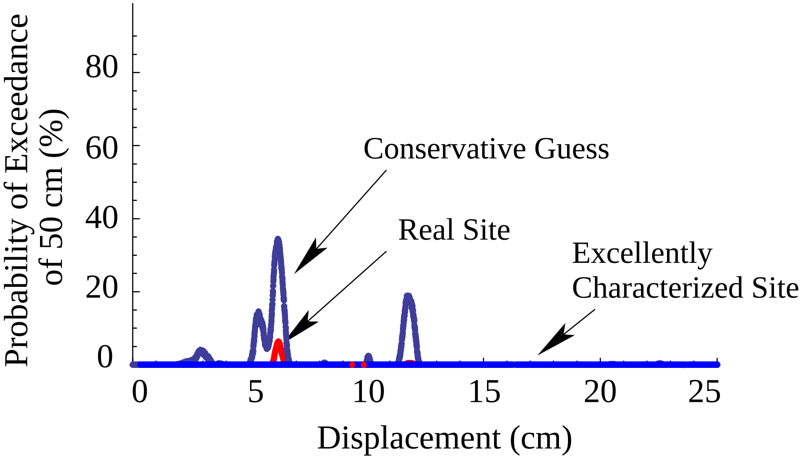
Example: CDF (Non-Exceedance) at 6 s



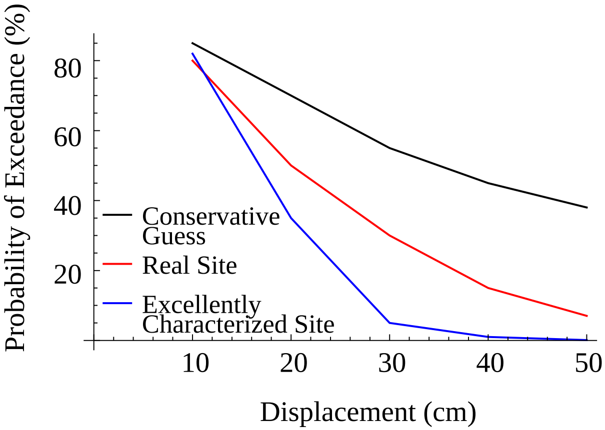
Probability of Exceedance of 20cm



Probability of Exceedance of 50cm



Risk of Unacceptable Deformation



Summary

- Behavior of all civil engineering objects (structures, soils...) is probably probabilistic!
- Presented methodology (PEP and SEPFEM) allows for (very) accurate numerical simulation of PDFs of DOFs (and stress, strain) from known (given) PDFs of material properties and PDFs of loads.
- Human nature: how much do you want to know about potential problem?